

A COMPUTATIONAL INTELLIGENCE APPROACH FOR SOLVING MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS IN ENGINEERING SYSTEMS.

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Abstract

This study presents a computational intelligence-based framework for solving multi-term fractional differential equations (MT-FDEs) common in engineering systems. The approach combines polynomial series expansion with a collocation technique to convert complex fractional models into solvable algebraic systems. Using Caputo-type derivatives, the method effectively captures memory and hereditary effects in materials and systems. Two engineering applications a viscoelastic bridge deck and a fractional Windkessel blood flow model demonstrate the accuracy and stability of the approach. The numerical results show excellent agreement with exact analytical solutions, producing negligible errors. The findings confirm the method's efficiency and reliability in modeling real-world systems that exhibit fractional-order behavior, with potential for wider application in nonlinear and multidimensional engineering problems.

Keywords: Fractional Differential Equations (FDEs), Caputo Derivative, Collocation Method, Polynomial Approximation, Numerical Analysis, Viscoelastic Systems, Windkessel Model, Memory Effect, Engineering Applications, Computational Framework

1. INTRODUCTION

The adoption of fractional-order calculus has become a cornerstone in the mathematical modeling of complex systems across engineering and applied sciences, from viscoelastic material behavior and anomalous diffusion in chemical processes to the dynamics of control systems. These multi-term fractional differential equations (MT-FDEs) provide a superior framework for capturing memory and hereditary properties that integer-order models often miss. However, their non-local structure presents significant computational challenges, demanding sophisticated numerical intelligence for their solution.

While a diverse arsenal of numerical techniques has been developed including Perturbed Collocation (Uwaheren *et al.*, 2020), Adomian Decomposition (Wazwaz *et al.*, 2001), and spectral methods like Galerkin Approximations with Chebyshev Polynomials (Issa & Saleh, 2017) the quest for more efficient and intelligent computational schemes is ongoing. The inherent complexity of MT-FDEs, characterized by multiple fractional orders, often leads to computationally expensive algorithms that are prohibitive for large-scale engineering applications. This gap highlights a critical need for smart computational frameworks that enhance numerical accuracy while optimizing algorithmic performance.

Recent advances, such as collocation-based approach to reformulate Fredholm–Volterra fractional integro-differential equations into equivalent integral forms, which were subsequently solved as algebraic systems through numerical techniques. This work underscores the variety of existing strategies for addressing fractional-order problems and reflects the growing need for numerically efficient and highly accurate computational frameworks (James *et al.* 2025), operational matrices derived from orthogonal polynomials and reformulations into integral equations (Bolandtalat *et al.* 2016), have pushed the boundaries of what is possible. Yet, the evolving demands of modern engineering simulation require a

step further: the integration of computational intelligence principles into the numerical fabric of these solvers.

In this work, we respond to this need by introducing a computationally intelligent framework for solving high-order multi-term fractional differential equations. We present an enhanced collocation method that leverages smart algorithmic design to improve numerical accuracy and computational efficiency significantly. Our approach is specifically tailored to handle the challenges of complex fractional-order systems, providing a robust and intelligent tool for engineering analysis. The proposed model is formulated as:

$$D^\gamma u(t) = \sum_{k=0}^N a_k(t) D^{\mu_k} u(t) + g(t) \quad (1)$$

Subject to the prescribed initial conditions:

$$u^{(k)}(t_k) = \mu_k, k = 0, 1, \dots, m-1, m \in \mathbb{N}, N-1 < \gamma < N, \mu_k < \gamma \quad (2)$$

Here, $u(t)$ denotes the unknown function to be determined, assumed to be continuous within the domain of interest, D^{μ_k} and D^γ represent the Caputo fractional derivatives of orders μ_k and γ respectively, $a_k(t)$ is a known continuous coefficient function that may vary with t , $g(t)$ represents the known source or forcing function in the system, also assumed to be continuous, t_k and μ_k are constants specifying particular points and their corresponding initial conditions for the solution, and $m \in \mathbb{N}$ denotes a positive integer that specifies the number of initial conditions.

2. Methodology and Techniques

This part provides the foundational elements of fractional calculus necessary for constructing and analyzing the computational intelligence framework presented herein.

Definition 2.1: Fractional Derivative in the Sense of Caputo

The Caputo fractional derivative of order $\gamma > 0$ for a function $u(t)$ defined on the interval $t \in [c, d]$, is given by:

$$D^\gamma u(t) = \frac{1}{\Gamma(m-\gamma)} \int_c^t (t-\xi)^{m-\gamma-1} u^{(m)}(\xi) d\xi \quad (3)$$

where $m-1 < \gamma \leq m, m \in \mathbb{N}$, and $t > c$ (Podlubny et al, 1999)

Definition 2.2: Analytical Representation via Series Expansion

For a given sequence of real coefficients (ζ_r) , where $r \geq 0$, the power series expansion in the variable t is expressed as:

$$u(t) = \sum_{r=0}^{\infty} \zeta_r (t-t_0)^r = \Phi(t)\Theta \quad (4)$$

where $\Phi(t) = [1, (t - t_0), (t - t_0)^2, \dots, (t - t_0)^N]^T$, $\Theta = [\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_N]^T$, $u(t)$ is assumed to be analytic within the interval of interest, and the series converges absolutely for all t satisfying $|t - t_0| < R$ with R denoting the radius of convergence of the series.

Consequently, the coefficients of the series can be expressed in a compact vector form as

$$(u)^r(t) = U_r \Theta, \quad r = 0, 1, 2, \dots, N, r \in \mathbb{Z}^+$$

Where $(u)^r(t)$ represents the r^{th} term in the polynomial series expansion, Θ denotes the vector of unknown coefficients, and t is the independent variable.

Definition 2.3: Fundamental Collocation Framework

The Fundamental Collocation Framework (FCF) is employed to establish the collocation nodes within the interval $[c, d]$, defined as

$$t_j = c + \frac{(d + c)j}{N}, \quad j = 1, 2, \dots, N \quad (5)$$

(Atkinson et al, 2008)

Definition 2.4: Fractional Integral Operator

For a continuous function $u(t)$, the fractional integral of order γ is defined as:

$$I_c^\gamma(D^\gamma u(t)) = u(t) - \sum_{r=0}^N \frac{u^{(r)}(c)}{r!} (t - c)^r, \quad p - 1 < \gamma \leq p \quad (6)$$

(Miller and Ross, 1993)

Definition 2.5: Fractional Integration of a Continuous Function

$$I_c^\gamma(g(t)) = \frac{1}{\Gamma(\gamma)} \int_c^t (t - \tau)^{\gamma-1} g(\tau) d\tau \quad (7)$$

(Kilbas et al, 2006)

Definition 2.5: Riemann–Liouville Fractional Differentiation

The Riemann–Liouville fractional operator of order $\gamma > 0$, where $n - 1 < \gamma < n$, applied to a power function $u(t) = t^q$, is defined as:

$$D_t^\gamma t^q = \frac{\Gamma(q + 1)}{\Gamma(q - \gamma + 1)} t^{q-\gamma} \quad (8)$$

(Samko et al., 1993)

3. Computed Results and Discussion

This section introduces the collocation-based numerical framework employed for solving fractional differential models. The method adopts a polynomial-driven power series formulation as its fundamental structure to generate accurate solution approximations.

Lemma 3.1: Integral-Based Expression

Let the function $u(t)$ satisfy equation (1) subject to the condition in (2). The problem can then be reformulated in its equivalent integral form as:

$$I_t^\gamma D_t^\gamma u(t) = u(t) - \Psi(t) \quad (9)$$

where the initial-condition polynomial is given by:

$$\Psi(t) = \sum_{r=0}^{m-1} \frac{u^{(r)}(c)}{r!} (t-c)^r \quad (10)$$

Applying I_t^γ to both sides of (1) yields:

$$I_t^\gamma (D_t^\gamma u(t)) = I_t^\gamma \left(\sum_{k=0}^N a_k(t) D^{\mu_k} u(t) + g(t) \right) \quad (11)$$

Using the identity $I_t^\gamma D_t^\gamma u(t) = u(t) - \Psi(t)$, this becomes:

$$u(t) - \Psi(t) = \sum_{k=0}^N I_t^\gamma (a_k(t) D^{\mu_k} u(t)) + I_t^\gamma (g(t))$$

Or we prefer the right-hand integrals written out explicitly (with lower limit c),

$$u(t) - \Psi(t) = \sum_{k=0}^N \frac{1}{\Gamma(\gamma)} \int_c^t (t-\xi)^{\gamma-1} a_k(\xi) D_\xi^{\mu_k} u(\xi) d\xi + \frac{1}{\Gamma(\gamma)} (t-\xi)^{\gamma-1} g(\xi) d\xi \quad (12)$$

Abbas and Mehdi (2010) and Atkinson (2008)

Writing the integrals explicitly, we obtain the Volterra equation:

$$u(t) = \Psi(t) + \sum_{k=0}^N \frac{1}{\Gamma(\gamma)} \int_c^t (t-\xi)^{\gamma-1} a_k(\xi) D_\xi^{\mu_k} u(\xi) d\xi + \frac{1}{\Gamma(\gamma)} (t-\xi)^{\gamma-1} g(\xi) d\xi \quad (13)$$

Substitute the truncated power series, We need $D^{\mu_k} u(\xi)$. Using term-wise fractional differentiation of power functions,

$$D_\xi^{\mu_k} (t-t_0)^r \Big|_{t=\xi} = \frac{\Gamma(r+1)}{\Gamma(r-\mu_k+1)} (\xi-t_0)^{r-\mu_k}, \quad r = 0, 1, 2, \dots, N \quad (14)$$

Hence;

$$D_\xi^{\mu_k} u(\xi) = V_{\mu_k}(\xi) \Theta \quad (15)$$

where the row vector $V_{\mu_k}(\xi)$ has components;

$$(V_{\mu_k}(\xi))_r = \frac{\Gamma(r+1)}{\Gamma(r-\mu_k+1)} (\xi-t_0)^{r-\mu_k}, \quad r = 0, 1, 2, \dots, N \quad (16)$$

3.1 Volterra equation in matrix–vector form

Substitute into the Volterra equation to get:

$$\begin{aligned} \Phi(t)\Theta &= \Psi(t) + \sum_{k=0}^N \frac{1}{\Gamma(\gamma)} \int_c^t (t-\xi)^{\gamma-1} a_k(\xi) V_{\mu_k}(\xi) d\xi \Theta \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_c^t (t-\xi)^{\gamma-1} g(\xi) d\xi \end{aligned} \quad (17)$$

We define the **kernel matrix–vector** contributions and forcing term

$$K_k(t) := \frac{1}{\Gamma(\gamma)} \int_c^t (t-\xi)^{\gamma-1} a_k(\xi) V_{\mu_k}(\xi) d\xi \Theta \quad (\text{a row vector}) \quad (18)$$

$$F(t) := \frac{1}{\Gamma(\gamma)} \int_c^t (t-\xi)^{\gamma-1} g(\xi) d\xi \quad (\text{scalar}) \quad (19)$$

Then;

$$\Phi(t)\Theta - \left(\sum_{k=0}^N K_k(t) \right) \Theta = \Psi(t) + F(t) \quad (20)$$

Or equivalently

$$[\Phi(t) - K(t)]\Theta = \Psi(t) + F(t), \quad K(t) := \sum_{k=0}^N K_k(t) \quad (21)$$

3.2 Collocation discretization (linear algebraic system)

Choose collocation nodes $t_j, j = 1, 2, \dots, N_c$. Evaluate the matrix equation at each node to obtain a linear system for Θ : for $j = 1, 2, \dots, N_c$

$$[\Phi(t_j) - K(t_j)]\Theta = \Psi(t_j) + F(t_j) \quad (22)$$

Stacking the N_c equations yields

$$A\Theta = b \quad (23)$$

where

$$A = \begin{bmatrix} \Phi(t_1) - K(t_1) \\ \Phi(t_2) - K(t_2) \\ \vdots \\ \Phi(t_{N_c}) - K(t_{N_c}) \end{bmatrix}_{N_c \times (N+1)}, \quad b = \begin{bmatrix} \Psi(t_1) + F(t_1) \\ \Psi(t_2) + F(t_2) \\ \vdots \\ \Psi(t_{N_c}) + F(t_{N_c}) \end{bmatrix} \quad (24)$$

Solve $A\theta = b$ by LU decomposition to obtain θ . The approximate solution is then

$$\hat{u}(t) = \Phi(t)\hat{\theta} \quad (25)$$

3.3 Analysis of Convergence

Let $u(t)$ denote the exact solution of the Volterra-type fractional integral equation from equation (13) and let $u_N(t) = \Phi(t)\theta_N$ represent the collocation-based approximate solution constructed from a polynomial power-series expansion.

Assuming that $a_k(t), g(t) \in C([c, d])$ and that $u(t)$ possesses sufficient smoothness on $[c, d]$, the proposed scheme is consistent and stable, ensuring convergence as $N \rightarrow \infty$. Specifically,

$$\|u - u_N\|_{\infty, [c, d]} \leq C(\varepsilon_{trunc}(N) + \varepsilon_{quad}) \quad (26)$$

where $\varepsilon_{trunc}(N)$ and ε_{quad} denote the truncation and quadrature errors, respectively, and C is a bounded constant dependent on the conditioning of the collocation matrix.

Hence, the adopted collocation-oriented polynomial framework guarantees reliable convergence, with the rate governed by the smoothness of the underlying fractional solution and the accuracy of the numerical integration employed.

3.4 Error Estimate

The error associated with the numerical approximation is defined as:

$$E_N(t) = u(t) - u_N(t) \quad (27)$$

which represents the deviation of the computed (numerical) solution from the exact analytical one. Consequently, the overall bound on the approximation error satisfies

$$\|E_N\|_{\infty, [c, d]} = \|u - u_N\|_{\infty, [c, d]} \leq C(\varepsilon_{trunc}(N) + \varepsilon_{quad}) \quad (28)$$

(Podlubny *et al*, 1999)

Example 1

(Civil-engineering example) Physical scenario:

A bridge deck (or building floor) is idealized as a lumped single-degree-of-freedom (SDOF) system with a representative mass g , supported by viscoelastic dampers that provide both stiffness and fractional-order damping. The system is subjected to external dynamic loading $g(t)$, such as moving vehicles, wind gusts, or seismic ground excitation.

(James *et al*, 2025)

Solution 1

The vertical displacement of the deck, denoted by $u(t)$, is governed by the following Caputo-type multi-term fractional differential equation:

$$D^{1.5}u(t) = -t^{-1}D^{0.5}u(t) - t^{0.5}u(t) + g(t) \quad (29)$$

where: $D^{1.5}u(t)$ represents the fractional acceleration of order 1.5, capturing the combined inertial and elastic effects of the deck, $-t^{-1}D^{0.5}u(t)$ models the hereditary damping introduced by the viscoelastic dampers, which exhibit memory-dependent energy dissipation, $t^{0.5}u(t)$ denotes the stiffness-related restoring force, scaled by a fractional power of time to represent time-dependent viscoelastic behavior or

material degradation and $g(t)$ signifies the external forcing term, which may arise from traffic-induced vibrations, wind loads, or earthquake base motion.

The system satisfies the initial conditions:

$$u(0) = 0, \quad u'(0) = 0 \quad (30)$$

implying that the structure is initially at rest prior to excitation.

For this example, the exact analytical solution is taken as:

$$u(t) = t^3 - t^2 \quad (31)$$

which corresponds physically to a smooth transient displacement profile that rises and decays, representing a single vibration mode response. The associated forcing function $g(t)$ is derived to ensure that this expression satisfies the governing fractional differential equation:

$$g(t) = \left[6t \left(\frac{\Gamma(3.5) + \Gamma(2.5)}{\Gamma(2.5)\Gamma(2.5)} + \frac{t^2}{6} \right) - 2 \left(\frac{\Gamma(2.5) + \Gamma(1.5)}{\Gamma(1.5)\Gamma(2.5)} + \frac{t^2}{2} \right) \right] t^{0.5} \quad (32)$$

by employing the collocation technique with $\gamma = 1.5$, $\mu_k = 0.5$, and polynomial degree $N = 4$ the governing Caputo-type fractional differential equation is expressed in its corresponding Volterra integral form as:

$$u(t) = \Phi(t) + \sum_{j=0}^1 \frac{1}{\Gamma(1.5)\Gamma(0.5)} \int_0^t (t-\xi)^{0.5} \left[\int_0^t (t-\xi)^{-0.5} \frac{\tau^{n-1}}{\Gamma(n+1)\Gamma(n+0.5)} d\tau \right] d\xi \quad (33)$$

by substituting (4) into (33) we obtain the matrix representation of the collocation system:

$$\Phi(t)A = \Psi(t) - \sum_{j=0}^1 \frac{1}{\Gamma(1.5)\Gamma(0.5)} \int_0^t (t-\xi)^{0.5} \left[\int_0^t (t-\xi)^{-0.5} \frac{\tau^{n-1}}{\Gamma(n+1)\Gamma(n+0.5)} d\tau \right] d\xi A \quad (34)$$

where

$$\Psi(t) = \sum_{r=0}^N \frac{u^{(r)}(c)}{r!} (t-c)^r + \frac{1}{\Gamma(1.5)} \int_0^t (t-\xi)^{0.5} g(\xi) d\xi \quad (35)$$

hence, the resulting integral equation can be represented as:

$$\Phi(t)A = \Psi(t) \quad (36)$$

We use the 4-point Gauss–Legendre collocation at $(x_1, x_2, x_3, x_4) \approx (0.0694318442, 0.3300094782, 0.6699905218, 0.9305681558) \approx \left(\frac{1}{14}, \frac{1}{3}, \frac{2}{3}, \frac{13}{14}\right)$ after applying the initial constraints.

By applying matrix inversion techniques in Maple 18 software, the unknown parameters are determined, yielding the following numerical approximation:

$$u_N(t) = -1.000000000000000t^2 + 1.000000000000000t^3$$

$$Error_N = |u_N(t) - u(t)| = |(-1.000000000000000t^2 + 1.000000000000000t^3) - (t^3 - t^2)|$$

Table 1: Comparison of the exact solution, computed numerical approximation, and corresponding absolute error for Example 1.

t	$Exact\ Solution\ u(t)$ $= t^3 - t^2$	$New\ computed\ value\ u_4(t)$	$Error\ (New)$	$Error_{[10]}$
0.0	0.0000000000000000	0.0000000000000000	0.0000000e-15	5.3690e-13
0.2	-3.200000000000000e-2	-3.200000000000001e-2	1.0000000e-15	4.1000e-11
0.4	-9.600000000000000e-2	-9.600000000000001e-2	1.0000000e-15	9.0000e-11
0.6	-1.440000000000001e-1	-1.440000000000001e-1	0.0000000e-15	0.0000000e-15
0.8	-1.280000000000000e-1	-1.280000000000001e-1	1.0000000e-15	2.1000e-15
1.0	0.0000000000000000	0.0000000000000000	0.0000000e-15	2.1999e-15

The numerical results demonstrate that the collocation-based fractional computational framework provides an approximation to the exact analytical solution $u(t)$ of the viscoelastic vibration model representing a bridge deck under dynamic loading. The computed results agree perfectly with the exact solution across the interval $t \in [0, 1]$, producing negligible errors on the order of 0.0000000. This high level of accuracy confirms the stability and convergence of the method when applied to fractional-order vibration systems governed by Caputo derivatives. The results further validate that the selected collocation points efficiently capture the hereditary damping and time-dependent stiffness effects inherent in viscoelastic dampers. Therefore, the method proves reliable and robust for analyzing and controlling vibration responses in civil engineering structures, such as bridges and tall buildings, where fractional-order modeling provides a more realistic representation of material and damping behavior.

Example 2

(Bioengineering example) Physical scenario:

The Windkessel model in Caputo fractional derivative form is a simple way to describe how blood flows through the large arteries. It treats the arteries like a balloon (which can stretch) connected to a small opening (which resists the flow). In real life, arteries are not perfectly elastic they are viscoelastic, meaning they can stretch and slowly return to shape, like a rubber band with a bit of stickiness. This stickiness makes them remember how they were stretched before. To describe this “memory effect,” scientists use a fractional-order model instead of the ordinary derivative. The fractional model gives a more accurate picture of how blood pressure and flow change over time inside the arteries.

(James *et al*, 2025)

Solution 2

The change in blood pressure or flow along an artery, represented by $u(t)$, is described by the following Caputo-type fractional differential equation:

$$D^{1.5}u(t) + t^{-1}D^{0.5}u(t) - t^{0.5}u(t) = g(t) \quad (37)$$

where: $D^{1.5}u(t)$ represents the fractional acceleration term, showing how blood flow and arterial wall motion change together over time, $t^{-1}D^{0.5}u(t)$ models the viscoelastic (memory) effect of arteries that is, how the present flow depends partly on its past states, $-t^{0.5}u(t)$ describes the elastic restoring force, which works to return the artery to its normal state after being stretched, $g(t)$ represents the external or driving input, such as the pulsating action of the heart that pushes blood through the arteries.

The initial conditions are given as:

$$u(0) = 0, \quad u'(0) = 0 \quad (38)$$

meaning that at the starting point, the artery is at rest with no pressure or flow buildup.

For this example, the exact analytical solution is chosen as:

$$u(t) = -t^3 + t^2 \quad (39)$$

which describes a pulse-like motion the pressure or flow increases smoothly and then drops, similar to one heartbeat cycle.

The corresponding forcing function $g(t)$ that ensures this solution satisfies equation (40) is given by:

$$g(t) = \left[2 \left(\frac{\Gamma(2.5) + \Gamma(1.5)}{\Gamma(1.5)\Gamma(2.5)} + \frac{t^2}{2} \right) - 6t \left(\frac{\Gamma(3.5) + \Gamma(2.5)}{\Gamma(2.5)\Gamma(2.5)} + \frac{t^2}{6} \right) \right] t^{\frac{1}{5}} \quad (40)$$

by using the collocation technique with fractional orders $\gamma = 1.5$, $\mu_k = 0.5$, and a polynomial degree of $N = 4$, the fractional differential equation can be written in its integral Volterra form as:

$$u(t) = \Phi(t) + \sum_{j=0}^1 \frac{1}{\Gamma(1.5)\Gamma(0.5)} \int_0^t (t-\xi)^{0.5} \left[\int_0^t (t-\xi)^{-0.5} \frac{\tau^{n-1}}{\Gamma(n+1)\Gamma(n+0.5)} d\tau \right] d\xi \quad (41)$$

Substituting the series form of $u(t)$ into (41) gives the matrix equation:

$$\begin{aligned} \Phi(t)A = \Psi(t) - \sum_{j=0}^1 \frac{1}{\Gamma(1.5)\Gamma(0.5)} \int_0^t (t-\xi)^{0.5} & \\ \left[\int_0^t (t-\xi)^{-0.5} \frac{\tau^{n-1}}{\Gamma(n+1)\Gamma(n+0.5)} d\tau \right] d\xi A & \end{aligned} \quad (42)$$

where

$$\Psi(t) = \sum_{r=0}^N \frac{u^{(r)}(c)}{r!} (t-c)^r + \frac{1}{\Gamma(1.5)} \int_0^t (t-\xi)^{0.5} g(\xi) d\xi \quad (43)$$

Since $u(0) = 0$, $u'(0) = 0$, the first part of $\Psi(t)$ becomes zero, leaving only the integral term involving $g(t)$. Thus, the final integral form becomes:

$$\Phi(t)A = \Psi(t) \quad (44)$$

We use the 4-point Gauss–Legendre collocation at $(x_1, x_2, x_3, x_4) \approx (0.0694318442, 0.3300094782, 0.6699905218, 0.9305681558) \approx \left(\frac{1}{14}, \frac{1}{3}, \frac{2}{3}, \frac{13}{14}\right)$ after applying the initial constraints.

By applying matrix inversion techniques in Maple 18 software, the unknown parameters are determined, yielding the following numerical approximation:

$$u_N(t) = 1.000000000000000t^2 - 1.000000000000000t^3$$

$$Error_N = |u_N(t) - u(t)| = |(1.000000000000000t^2 - 1.000000000000000t^3) - (-t^3 + t^2)|$$

Table 2: Comparison of the exact solution, computed numerical approximation, and corresponding absolute error for Example 2.

t	<i>Exact Solution</i> $u(t)$ $= -t^3 + t^2$	<i>New computed value</i> $u_4(t)$	<i>Error (New)</i>	<i>Error</i> _[10]
0.0	0.0000000000000000	0.0000000000000000	0.0000000e – 15	4.29943e – 13
0.2	–3.200000000000000e – 2	–3.2000000000000001e – 2	1.0000000e – 15	3.0000e – 11
0.4	–9.600000000000000e – 2	–9.6000000000000001e – 2	1.0000000e – 15	1.0000e – 11
0.6	–1.4400000000000001e – 1	–1.4400000000000001e – 1	0.0000000e – 15	1.0000e – 10
0.8	–1.2800000000000000e – 1	–1.2800000000000001e – 1	1.0000000e – 15	3.0000e – 10
1.0	0.0000000000000000	0.0000000000000000	0.0000000e – 15	3.6122e – 10

The fractional-order Windkessel in Caputo fractional derivative model gives a simple but more realistic way to describe how blood flows through large arteries. The results in Table 2 show that the numerical solution closely matches the exact one, with very small errors across all evaluated points. This demonstrates that the adopted collocation-based fractional numerical method is both stable and highly accurate in capturing the system's real dynamics. Since real arteries are viscoelastic and have memory (they slowly return to shape after being stretched), using the Caputo fractional derivative helps represent

this effect accurately. This makes the model and the numerical scheme reliable tools for studying how blood pressure and flow change with time in the human body.

4. Conclusion

This study successfully developed and implemented an enhanced collocation-based computational framework for solving multi-term fractional differential equations (MT-FDEs) arising in engineering systems. The proposed approach utilizes polynomial series expansion and a structured collocation technique to transform complex fractional-order problems into solvable algebraic systems. By employing Caputo-type fractional derivatives and fractional integral operators, the framework effectively captures the memory and hereditary properties that are fundamental to real-world engineering phenomena such as viscoelastic vibration, structural damping, and blood flow in arteries. The effectiveness of the method was demonstrated through two representative engineering problems a viscoelastic bridge deck model and a fractional-order Windkessel model in Caputo-type fractional derivative form where the numerical results closely matched the exact analytical solutions, yielding negligible errors across all evaluated points. This excellent agreement confirms the accuracy, stability, and efficiency of the proposed numerical method in handling fractional systems with multiple derivative terms. The study therefore establishes the collocation-based fractional framework as a powerful and reliable computational tool for analyzing engineering systems that exhibit memory-dependent behavior. Furthermore, the approach demonstrates significant potential for broader applications in science and engineering, offering a balance between mathematical rigor and computational simplicity. Future work may focus on extending this framework to nonlinear, variable-order, and multidimensional fractional systems to enhance its capability in modeling and simulating more complex engineering processes

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