

NEW HYBRID IMPLICIT QUARTER-STEP BLOCK METHOD FOR THE SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATION

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Abstract

This paper introduces an innovative hybrid implicit quarter-step block method designed for solving first-order ordinary differential equations (ODEs) numerically. Traditional linear multistep methods often necessitate the creation of separate predictors, which can ramp up both computational costs and complexity. To tackle this issue, we propose a continuous implicit hybrid method that utilizes interpolation and collocation of a power series approximation at both grid and off-grid points within a single integration interval. By evaluating the continuous formulation at quarter-step points, we derive discrete block formulas that come together to create a unified, self-starting block method. This approach allows for the simultaneous generation of numerical approximations at four consecutive points, eliminating the need for a separate predictor. We've established that this method is zero-stable, consistent, and therefore convergent. To evaluate the effectiveness of our proposed method, we conducted numerical experiments across a variety of initial value problems. The findings reveal that our new quarter-step block method not only achieves greater accuracy but also outperforms existing methods of similar order, highlighting its potential as a powerful and efficient computational tool for integrating first-order ODEs.

Keywords: Block Method; Quarter-Step; Hybrid Method; Ordinary Differential Equations; Numerical Solution; Implicit Scheme; Convergence Analysis.

1. Introduction

Ordinary differential equations (ODEs) are essential for modeling a wide range of physical, biological, and engineering systems. First-order ODEs, in particular, arise in various fields, from population dynamics to electrical circuits and chemical kinetics. While there are sufficient numerical techniques available for solving these equations, the ongoing search for more efficient and accurate methods continues to be a large topic in research.

Traditional numerical methods, like single-step techniques such as Runge-Kutta and multistep methods like Adams-Bashforth-Moulton, achieve varying trade-offs between accuracy and computational cost. However, these methods can face challenges related to stability and stiffness, especially when tackling certain real-world problems [1]. Block methods have emerged as a solution to enhance parallelizability and computational efficiency, gaining traction for their ability to process multiple points at once [2]. The hybrid implicit quarter-step method is an exciting new development aimed at boosting numerical accuracy and stability without sacrificing computational efficiency. This quarter-step technique sharpens solution approximations by assessing intermediate points at shorter intervals, which not only improves accuracy but also cuts down on local truncation errors when compared to traditional half-step methods [3], [4]. By adding more sub-interval calculations, this method achieves a higher level of precision while still keeping computational efficiency in check.

In this paper, we introduce a new hybrid implicit quarter-step block method specifically designed for the numerical solution of first-order ODEs. This method merges the efficiency of block methods with the reliability of implicit schemes and the enhanced accuracy provided by quarter-step formulations. We present a theoretical

analysis of its stability and convergence, along with numerical experiments that showcase its advantages over some existing methods. We consider the numerical solution of first-order initial value problems of the form:

$$y' = f(x, y), \quad a \leq x \leq b \quad y(x_0) = y_0 \quad (1.1)$$

Many researchers have played a key role in advancing continuous linear multistep methods for solving differential equations. Notable contributions come from scholars like [5], [6], [7], [8], and [9]. These experts have introduced a variety of techniques, including predictor-corrector methods and discrete block methods.

2. Derivation of the Block Hybrid Method

In this section, we will utilize the concepts introduced in preceding sections to construct block hybrid method aimed at solving first-order initial value problems in differential equation expressed in the form (1.1). The power series as an approximate solution of the form:

$$y(x) = \sum_{j=0}^{s+r-1} \alpha_j x^j \quad (2.1)$$

2.2 Formulation of the Block Hybrid Method

The power series polynomial (2.1) is consider as an approximate solution of (1.1).

Differentiate (2.1) once to yield,

$$y'(x) = h \sum_{j=0}^{s+r-1} j \alpha_j x^{j-1} \quad (2.2)$$

Where $\alpha \in \mathbb{R}$ for $j = 0 \left(\frac{1}{28}\right) \frac{1}{4}$ and $y(x)$ is continuously differential. Let the solution of (1.1) be sought on the integration interval $[a, b]$ with a constant h step-size defined by $h = x_j - x_{j-1}, j = 0, 1, 2, \dots, N$, substituting equation (2.2) into (1.1) gives,

$$f(x, y) = h \sum_{j=0}^{s+r-1} j \alpha_j x^{j-1} \quad (2.3)$$

We interpolate (2.1) at $x_{n+r}, r = 0$ and Collocate (2.3) at $x_{n+s}, s = 0, \frac{1}{28}, \frac{1}{14}, \frac{3}{28}, \frac{1}{7}, \frac{5}{28}, \frac{3}{14}, \frac{1}{4}$, gives a system of linear equation in the form:

$$AX = U \quad (2.4)$$

Where

$$A = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad a_9]^T$$

$$U = \left[y_n, f_n, f_{n+\frac{1}{28}}, f_{n+\frac{1}{14}}, f_{n+\frac{3}{28}}, f_{n+\frac{1}{7}}, f_{n+\frac{5}{28}}, f_{n+\frac{3}{14}}, f_{n+\frac{1}{4}} \right]^T$$

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 \\ 0 & 1 & 2x_{n+\frac{1}{28}} & 3x_{n+\frac{1}{28}}^2 & 4x_{n+\frac{1}{28}}^3 & 5x_{n+\frac{1}{28}}^4 & 6x_{n+\frac{1}{28}}^5 & 7x_{n+\frac{1}{28}}^6 & 8x_{n+\frac{1}{28}}^7 \\ 0 & 1 & 2x_{n+\frac{1}{14}} & 3x_{n+\frac{1}{14}}^2 & 4x_{n+\frac{1}{14}}^3 & 5x_{n+\frac{1}{14}}^4 & 6x_{n+\frac{1}{14}}^5 & 7x_{n+\frac{1}{14}}^6 & 8x_{n+\frac{1}{14}}^7 \\ 0 & 1 & 2x_{n+\frac{3}{28}} & 3x_{n+\frac{3}{28}}^2 & 4x_{n+\frac{3}{28}}^3 & 5x_{n+\frac{3}{28}}^4 & 6x_{n+\frac{3}{28}}^5 & 7x_{n+\frac{3}{28}}^6 & 8x_{n+\frac{3}{28}}^7 \\ 0 & 1 & 2x_{n+\frac{1}{7}} & 3x_{n+\frac{1}{7}}^2 & 4x_{n+\frac{1}{7}}^3 & 5x_{n+\frac{1}{7}}^4 & 6x_{n+\frac{1}{7}}^5 & 7x_{n+\frac{1}{7}}^6 & 8x_{n+\frac{1}{7}}^7 \\ 0 & 1 & 2x_{n+\frac{5}{28}} & 3x_{n+\frac{5}{28}}^2 & 4x_{n+\frac{5}{28}}^3 & 5x_{n+\frac{5}{28}}^4 & 6x_{n+\frac{5}{28}}^5 & 7x_{n+\frac{5}{28}}^6 & 8x_{n+\frac{5}{28}}^7 \\ 0 & 1 & 2x_{n+\frac{3}{14}} & 3x_{n+\frac{3}{14}}^2 & 4x_{n+\frac{3}{14}}^3 & 5x_{n+\frac{3}{14}}^4 & 6x_{n+\frac{3}{14}}^5 & 7x_{n+\frac{3}{14}}^6 & 8x_{n+\frac{3}{14}}^7 \\ 0 & 1 & 2x_{n+\frac{1}{4}} & 3x_{n+\frac{1}{4}}^2 & 4x_{n+\frac{1}{4}}^3 & 5x_{n+\frac{1}{4}}^4 & 6x_{n+\frac{1}{4}}^5 & 7x_{n+\frac{1}{4}}^6 & 8x_{n+\frac{1}{4}}^7 \end{bmatrix}$$

Solving α_j s in (2.4) and substituting back into (2.1) gives a linear block scheme as:

$$y(t) = \alpha_0(t)y_0 + h \left[\beta_0(t)f_0 + \beta_{\frac{1}{28}}(t)f_{\frac{1}{28}} + \beta_{\frac{1}{14}}(t)f_{\frac{1}{14}} + \beta_{\frac{3}{28}}(t)f_{\frac{3}{28}} + \beta_{\frac{1}{7}}(t)f_{\frac{1}{7}} + \beta_{\frac{5}{28}}(t)f_{\frac{5}{28}} + \beta_{\frac{3}{14}}(t)f_{\frac{3}{14}} + \beta_{\frac{1}{4}}(t)f_{\frac{1}{4}} \right] \quad 2.5$$

Where $\alpha_0 = 1$

$$\beta_0 = -\frac{1}{270}(90354432t^8 - 103262208t^7 + 49479808t^6 - 12907776t^5 + 1990086t^4 - 183848t^3 + 9801t^2 - 270t)$$

$$\beta_{\frac{1}{28}} = \frac{1}{135}(316240512t^8 - 348509952t^7 + 158658080t^6 - 38377584t^5 + 5252016t^4 - 393372t^3 + 13230t^2)$$

$$\beta_{\frac{1}{14}} = -\frac{1}{15}(105413504t^8 - 111867392t^7 + 48404160t^6 - 10910144t^5 + 1347647t^4 - 86142t^3 + 2205t^2)$$

$$\beta_{\frac{3}{28}} = \frac{1}{135}(1581202560t^8 - 1613472000t^7 + 664212640t^6 - 140487312t^5 + 16011240t^4 - 930020t^3 + 22050t^2)$$

$$\beta_{\frac{1}{7}} = -\frac{1}{270}(3162405120t^8 - 3097866240t^7 + 1215482240t^6 - 243403776t^5 + 26188050t^4 - 1446480t^3 + 33075t^2)$$

$$\beta_{\frac{5}{28}} = \frac{1}{15}(105413504t^8 - 98959616t^7 + 37109856t^6 - 7106960t^5 + 735392t^4 - 39396t^3 + 882t^2)$$

$$\beta_{\frac{3}{14}} = -\frac{1}{135}(316240512t^8 - 283971072t^7 + 102186560t^6 - 18900672t^5 + 1902621t^4 - 99862t^3 + 2205t^2)$$

$$\beta_{\frac{1}{4}} = \frac{1}{135}(45177216t^8 - 38723328t^7 + 13445600t^6 - 2420208t^5 + 238728t^4 - 12348t^3 + 270t^2)$$

Where $t = \frac{x-x_n}{h}$.

Solving (2.5) for the independent solution gives a continuous block method in the form:

$$y_{n+k} = \sum_{j=0}^{\mu-1} \frac{(jh)^m}{m!} y_n^{(m)} + h^\mu \sum_{j=0}^s \sigma_j(x) f_{n+j} \quad 2.6$$

Where μ is the order of the differential equation, s is the collocation points. Hence the coefficient of f_{n+j} in (2.6)

$$\begin{aligned} \sigma_0 &= -\frac{1}{270}(90354432t^8 - 103262208t^7 + 49479808t^6 - 12907776t^5 + 1990086t^4 - 183848t^3 \\ &\quad + 9801t^2 - 270t) \\ \sigma_{\frac{1}{28}} &= \frac{1}{135}(316240512t^8 - 348509952t^7 + 158658080t^6 - 38377584t^5 + 5252016t^4 - 393372t^3 \\ &\quad + 13230t^2) \\ \sigma_{\frac{1}{14}} &= -\frac{1}{15}(105413504t^8 - 111867392t^7 + 48404160t^6 - 10910144t^5 + 1347647t^4 - 86142t^3 \\ &\quad + 2205t^2) \\ \sigma_{\frac{3}{28}} &= \frac{1}{135}(1581202560t^8 - 1613472000t^7 + 664212640t^6 - 140487312t^5 + 16011240t^4 \\ &\quad - 930020t^3 + 22050t^2) \\ \sigma_{\frac{1}{7}} &= -\frac{1}{270}(3162405120t^8 - 3097866240t^7 + 1215482240t^6 - 243403776t^5 + 26188050t^4 \\ &\quad - 1446480t^3 + 33075t^2) \\ \sigma_{\frac{5}{28}} &= \frac{1}{15}(105413504t^8 - 98959616t^7 + 37109856t^6 - 7106960t^5 + 735392t^4 - 39396t^3 + 882t^2) \\ \sigma_{\frac{3}{14}} &= -\frac{1}{135}(316240512t^8 - 283971072t^7 + 102186560t^6 - 18900672t^5 + 1902621t^4 - 99862t^3 \\ &\quad + 2205t^2) \\ \sigma_{\frac{1}{4}} &= \frac{1}{135}(45177216t^8 - 38723328t^7 + 13445600t^6 - 2420208t^5 + 238728t^4 - 12348t^3 + 270t^2) \end{aligned}$$

Where $t = \frac{x-x_n}{h}$.

Evaluating (2.6) at $t = 0 \left(\frac{1}{28} \right) \frac{1}{4}$ gives a discrete block formula of the form

$$Y_m = ey_n + hdf(y_n) + hbf(Y_m) \quad 2.7$$

Where e, d are $r \times r$ matrix

$$\text{Hence } d = \left[\frac{751}{69120} \quad \frac{41}{3920} \quad \frac{265}{25088} \quad \frac{139}{13230} \quad \frac{265}{25088} \quad \frac{41}{3920} \quad \frac{751}{69120} \right]^T$$

$$Y_m = \left[y_{n+\frac{1}{28}} \quad y_{n+\frac{1}{14}} \quad y_{n+\frac{3}{28}} \quad y_{n+\frac{1}{7}} \quad y_{n+\frac{5}{28}} \quad y_{n+\frac{3}{14}} \quad y_{n+\frac{1}{4}} \right]^T$$

$$e = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} \frac{139849}{3386880} & -\frac{4511}{125440} & \frac{123133}{3386880} & -\frac{88547}{3386880} & \frac{1537}{125440} & -\frac{11351}{3386880} & \frac{275}{677376} \\ \frac{733}{13230} & -\frac{71}{11760} & \frac{17}{735} & -\frac{1927}{105840} & \frac{13}{1470} & -\frac{29}{11760} & \frac{2}{6615} \\ \frac{1359}{1359} & \frac{1377}{1377} & \frac{5927}{5927} & -\frac{3033}{3033} & \frac{1377}{1377} & -\frac{373}{373} & \frac{9}{9} \\ \frac{25088}{362} & \frac{125440}{2} & \frac{125440}{446} & -\frac{125440}{53} & \frac{125440}{2} & -\frac{125440}{16} & \frac{25088}{2} \\ \frac{6615}{36725} & \frac{245}{775} & \frac{6615}{4625} & -\frac{13230}{13625} & \frac{245}{1895} & -\frac{6615}{275} & \frac{6615}{275} \\ \frac{677376}{27} & \frac{75264}{27} & \frac{75264}{17} & -\frac{677376}{27} & \frac{75264}{27} & -\frac{75264}{41} & \frac{677376}{0} \\ \frac{490}{3577} & \frac{3920}{49} & \frac{245}{2989} & -\frac{3920}{2989} & \frac{490}{49} & -\frac{3920}{3577} & \frac{0}{751} \\ \frac{69120}{69120} & \frac{2560}{2560} & \frac{69120}{69120} & -\frac{69120}{69120} & \frac{2560}{2560} & -\frac{69120}{69120} & \frac{69120}{69120} \end{bmatrix}$$

3. Analysis of Basic Properties of the Block Hybrid Method

The necessary and sufficient conditions for new method and their associated block method are analyzed to establish their validity. These properties include; order and error constant, consistency and zero-stability.

3.1 Order and Error Constant

This subsection establishes the linear operator $\phi[y(x_j); h]$ associated with the newly derived method.

Definition 3.1

The block linear multi step method of first order ODEs is said to be of order p if $c_0 = c_1 = c_2 = \dots = c_p$ and $c_{p+1} \neq 0$. Thus c 's are the coefficient of h and y function, while $c_{p+1} \neq 0$ is called the error constant.

Using Taylor series expansion we have

$$y_{n+\frac{1}{4}} = y_n + h \left[\frac{751}{69120} f_n + \frac{3577}{69120} f_{n+\frac{1}{28}} + \frac{49}{2560} f_{n+\frac{1}{14}} + \frac{2989}{69120} f_{n+\frac{3}{28}} + \frac{2989}{69120} f_{n+\frac{1}{7}} + \frac{49}{2560} f_{n+\frac{5}{28}} + \frac{3577}{69120} f_{n+\frac{3}{14}} + \frac{751}{69120} f_{n+\frac{1}{4}} \right]$$

$$\begin{pmatrix} 1 \\ \left(\frac{1}{4}\right)^1 \\ 1! \\ \left(\frac{1}{4}\right)^2 \\ 2! \\ \left(\frac{1}{4}\right)^3 \\ 3! \\ \left(\frac{1}{4}\right)^4 \\ 4! \\ \left(\frac{1}{4}\right)^5 \\ 5! \\ \left(\frac{1}{4}\right)^6 \\ 6! \\ \left(\frac{1}{4}\right)^7 \\ 7! \\ \left(\frac{1}{4}\right)^8 \\ 8! \\ \left(\frac{1}{4}\right)^9 \\ 9! \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{751}{69120} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{3577}{69120} \begin{pmatrix} 0 \\ 1 \\ \left(\frac{1}{28}\right)^1 \\ 1! \\ \left(\frac{1}{28}\right)^2 \\ 2! \\ \left(\frac{1}{28}\right)^3 \\ 3! \\ \left(\frac{1}{28}\right)^4 \\ 4! \\ \left(\frac{1}{28}\right)^5 \\ 5! \\ \left(\frac{1}{28}\right)^6 \\ 6! \\ \left(\frac{1}{28}\right)^7 \\ 7! \\ \left(\frac{1}{28}\right)^8 \\ 8! \end{pmatrix} - \frac{49}{2560} \begin{pmatrix} 0 \\ 1 \\ \left(\frac{1}{14}\right)^1 \\ 1! \\ \left(\frac{1}{14}\right)^2 \\ 2! \\ \left(\frac{1}{14}\right)^3 \\ 3! \\ \left(\frac{1}{14}\right)^4 \\ 4! \\ \left(\frac{1}{14}\right)^5 \\ 5! \\ \left(\frac{1}{14}\right)^6 \\ 6! \\ \left(\frac{1}{14}\right)^7 \\ 7! \\ \left(\frac{1}{14}\right)^8 \\ 8! \end{pmatrix} - \frac{2989}{69120} \begin{pmatrix} 0 \\ 1 \\ \left(\frac{3}{28}\right)^1 \\ 1! \\ \left(\frac{3}{28}\right)^2 \\ 2! \\ \left(\frac{3}{28}\right)^3 \\ 3! \\ \left(\frac{3}{28}\right)^4 \\ 4! \\ \left(\frac{3}{28}\right)^5 \\ 5! \\ \left(\frac{3}{28}\right)^6 \\ 6! \\ \left(\frac{3}{28}\right)^7 \\ 7! \\ \left(\frac{3}{28}\right)^8 \\ 8! \end{pmatrix} - \frac{2989}{69120} \begin{pmatrix} 0 \\ 1 \\ \left(\frac{1}{7}\right)^1 \\ 1! \\ \left(\frac{1}{7}\right)^2 \\ 2! \\ \left(\frac{1}{7}\right)^3 \\ 3! \\ \left(\frac{1}{7}\right)^4 \\ 4! \\ \left(\frac{1}{7}\right)^5 \\ 5! \\ \left(\frac{1}{7}\right)^6 \\ 6! \\ \left(\frac{1}{7}\right)^7 \\ 7! \\ \left(\frac{1}{7}\right)^8 \\ 8! \end{pmatrix} - \frac{49}{2560} \begin{pmatrix} 0 \\ 1 \\ \left(\frac{5}{28}\right)^1 \\ 1! \\ \left(\frac{5}{28}\right)^2 \\ 2! \\ \left(\frac{5}{28}\right)^3 \\ 3! \\ \left(\frac{5}{28}\right)^4 \\ 4! \\ \left(\frac{5}{28}\right)^5 \\ 5! \\ \left(\frac{5}{28}\right)^6 \\ 6! \\ \left(\frac{5}{28}\right)^7 \\ 7! \\ \left(\frac{5}{28}\right)^8 \\ 8! \end{pmatrix} - \frac{3577}{69120} \begin{pmatrix} 0 \\ 1 \\ \left(\frac{3}{14}\right)^1 \\ 1! \\ \left(\frac{3}{14}\right)^2 \\ 2! \\ \left(\frac{3}{14}\right)^3 \\ 3! \\ \left(\frac{3}{14}\right)^4 \\ 4! \\ \left(\frac{3}{14}\right)^5 \\ 5! \\ \left(\frac{3}{14}\right)^6 \\ 6! \\ \left(\frac{3}{14}\right)^7 \\ 7! \\ \left(\frac{3}{14}\right)^8 \\ 8! \end{pmatrix} - \frac{751}{69120} \begin{pmatrix} 0 \\ 1 \\ \left(\frac{1}{4}\right)^1 \\ 1! \\ \left(\frac{1}{4}\right)^2 \\ 2! \\ \left(\frac{1}{4}\right)^3 \\ 3! \\ \left(\frac{1}{4}\right)^4 \\ 4! \\ \left(\frac{1}{4}\right)^5 \\ 5! \\ \left(\frac{1}{4}\right)^6 \\ 6! \\ \left(\frac{1}{4}\right)^7 \\ 7! \\ \left(\frac{1}{4}\right)^8 \\ 8! \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 167 \\ 111915746249932800 \end{pmatrix}$$

The order of the method is 9 and error constant is $-\frac{167}{111915746249932800}$. It also follows that $y_{n+\frac{1}{28}}, y_{n+\frac{1}{14}}, y_{n+\frac{3}{28}}, y_{n+\frac{1}{7}}, y_{n+\frac{5}{28}}$ and $y_{n+\frac{3}{14}}$ are of order $p = 9 > 1$ with the following error constant in table 3.1 below:

Table 3.1 Order and Error Constant of the new Method

Scheme	Order	Error constant
$y_{n+\frac{1}{28}}$	9	$-\frac{33953}{38387100963726950400}$
$y_{n+\frac{1}{14}}$	9	$-\frac{24481569492172800}{17}$
$y_{n+\frac{3}{28}}$	9	$-\frac{47391482671267400}{369}$
$y_{n+\frac{1}{7}}$	9	$-\frac{149949613139588400}{107}$
$y_{n+\frac{5}{28}}$	9	$-\frac{31336408949981184}{5}$
$y_{n+\frac{3}{14}}$	9	$-\frac{14809838334771200}{9}$
$y_{n+\frac{1}{4}}$	9	$-\frac{111915746249932800}{167}$

3.2 Consistency of the Method [7]

The block method of linear multi step method is said to consistent if it has order $p = 9 > 1$, hence the method is consistent.

3.3 Zero Stability of the Method

$$\bar{p}(\lambda) = \det[\lambda A^{(1)} - A^{(0)}] = 0$$

Where

$$\lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda - 1 \end{pmatrix}$$

Determinant: $\lambda^7 - \lambda^6 = \lambda^6(\lambda - 1)$

Therefore, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0, \lambda_7 = 1$. Therefore, the method is zero stable [8]

Dahlquist's theorem states that, the new method is convergent and consistency and zero-stability are analyzed and fulfilled.

3.4 Convergence

Theorem 3.1

Consistency and zero-stability are both required and sufficient conditions for a linear multistep method to be convergent. Therefore, the new method is convergent since it is consistent and zero-stable [9].

4 Results and Discussions

This section presents and discusses the results derived from various numerical examples. Additionally, the effectiveness of the proposed method is evaluated using three differential equations. For each case, the approximate solutions are compared to numerical benchmarks, and the absolute errors from the new method are contrasted with those from existing approaches to assess its accuracy and performance.

4.2 Numerical Examples

To evaluate the effectiveness of the developed methods, we present several numerical examples.

Problem 1

We consider a linear first ordinary differential equation

$$y' = -1, y(0) = 1, 0 \leq x \leq 1, h = 0.1$$

Exact solution $y(x) = e^{-x}$

This problem was solved by [10] using a block method of order seven. The results are shown in Table 1.

Table 1 for problem 1

X-value	Exact solution	Computed solution	Error	Error in [10]
0.1000	0.9048374180359595	0.9048374180359640	4.440892×10^{-15}	1.95961×10^{-11}
0.2000	0.8187307530779819	0.8187307530779899	7.993606×10^{-15}	4.81316×10^{-11}
0.3000	0.7408182206817179	0.7408182206817289	1.099121×10^{-14}	4.81316×10^{-11}
0.4000	0.6703200460356392	0.6703200460356525	1.332268×10^{-14}	5.80682×10^{-11}
0.5000	0.6065306597126333	0.6065306597126484	1.509903×10^{-14}	6.56779×10^{-11}

0.6000	0.5488116360940263	0.5488116360940427	1.643130×10^{-14}	7.13132×10^{-11}
0.7000	0.4965853037914094	0.4965853037914269	1.748601×10^{-14}	7.52815×10^{-11}
0.8000	0.4493289641172215	0.4493289641172394	1.798561×10^{-14}	7.78485×10^{-11}
0.9000	0.4065696597405989	0.4065696597406173	1.831868×10^{-14}	7.92453×10^{-11}
1.0000	0.3678794411714422	0.3678794411714605	1.837419×10^{-14}	7.96713×10^{-11}

Problem 2

We consider a linear first-order ordinary differential equation:

$$y' = x - y, y(0) = 1, 0 \leq x \leq 1, h = 0.1$$

Exact solution: $y(x) = x + e^{-x} - 1$.

This problem was solved by [10] using a block method of order seven. The results are shown in Table 2.

Table 2 for problem 2

X-value	Exact solution	Computed solution	Error	Error in [10]
0.1000	0.0048374180359596	0.0048374180359641	4.447831×10^{-15}	1.9595×10^{-11}
0.2000	0.0187307530779819	0.0187307530779900	8.076873×10^{-15}	3.54623×10^{-11}
0.3000	0.0408182206817178	0.0408182206817289	1.106060×10^{-14}	4.81315×10^{-11}
0.4000	0.0703200460356392	0.0703200460356526	1.333655×10^{-14}	5.80680×10^{-11}
0.5000	0.1065306597126334	0.1065306597126485	1.502964×10^{-14}	6.56779×10^{-11}
0.6000	0.1488116360940266	0.1488116360940428	1.618150×10^{-14}	7.13132×10^{-11}
0.7000	0.1965853037914096	0.1965853037914268	1.718070×10^{-14}	7.52814×10^{-11}
0.8000	0.2493289641172218	0.2493289641172395	1.762479×10^{-14}	7.78485×10^{-11}
0.9000	0.3065696597405994	0.3065696597406174	1.804112×10^{-14}	7.92403×10^{-11}
1.0000	0.3678794411714426	0.3678794411714607	1.815215×10^{-14}	7.96712×10^{-11}

Problem 3

We consider a first-order ordinary differential equation:

$$y' = x * y, y(0) = 1, 0 \leq x \leq 1, h = 0.1$$

Exact solution: $y(x) = e^{\frac{1}{2}x^2}$

This problem was solved by [10] using a block method of order seven. The results are shown in Table 3.

Table 3 for problem 3

X-value	Exact solution	Computed solution	Error	Error in [10]
0.1000	1.0050125208594010	1.0050125208593967	4.218847×10^{-15}	2.6067×10^{-11}
0.2000	1.0202013400267558	1.0202013400267389	1.687539×10^{-14}	8.4790×10^{-11}
0.3000	1.0460278599087169	1.0460278599086765	4.041212×10^{-14}	1.8684×10^{-11}
0.4000	1.0832870676749586	1.0832870676748780	8.060219×10^{-14}	3.5701×10^{-10}
0.5000	1.1331484530668263	1.1331484530666809	1.454392×10^{-13}	6.1054×10^{-9}
0.6000	1.1972173631218104	1.1972173631215617	2.486900×10^{-13}	1.0157×10^{-9}
0.7000	1.2776213132048868	1.2776213132044756	4.112266×10^{-13}	1.6445×10^{-9}
0.8000	1.3771277643359574	1.3771277643352922	6.652456×10^{-13}	2.6158×10^{-9}
0.9000	1.4993025000567675	1.4993025000557072	1.060263×10^{-12}	4.1110×10^{-9}
1.0000	1.6487212707001289	1.6487212706984566	1.672218×10^{-12}	6.4070×10^{-9}

5. Discussion of Results

We considered three numerical examples in this paper. The three examples were earlier solved by [10], where they applied an order seven block method. We applied a new order nine quarter-step hybrid block method to solve these three problems and from the results obtained, the quarter-step method performed better than the existing method with which we compared our results.

6. Conclusion

We have proposed an order nine hybrid implicit quarter-step method for the solution of first order ordinary differential equations. Our method was found to be zero stable, consistent and converges. The numerical examples show that our method gave better accuracy than the existing method.

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