e-ISSN: 3027-0650
Vol. 2, Issue 1, pp.51-55, November 6-9, 2024

TOPOLOGY AND TOPOLOGICAL SPACE IN FINITE GEOMETRY WITH VARIABLES IN Z_d

Semiu Oladipupo Oladejo[1], and Aliyu Muhammed Awwal[1]

[1]Department of Mathematics, Faculty of Science, Gombe State University, Gombe Nigeria
Email: {sooladejo@gsu.edu.ng, abdsemiu@yahoo.com and aliyuma@gsu.edu.ng[2]}

Abstract

In this paper, partial ordered relation is researched, finite geometry and its subgeometries are explored. An investigation of topology which exists in near-linear and non-near-linear finite geometry G_d is delved into. The outcome of this work shows an existence of the concept of topology and topological space on non-near-linear finite geometry with variables in Z_d. The complexity shown in this work demonstrated the existence of relationship between a geometry as a structure and its subgeometries as its substructures.

Keywords: Topology, ring of integers modulo d, lines, finite geometry.

1. INTRODUCTION

Let Z^+represents a set of positive integers. Z_d, the ring of integers modulo d, where $d \in Z^+$. For quite some time, finite quantum systems with variables in Z_d had received enormous attention with a special focus on mutually unbiased bases. Likewise in recent times, the weak mutually unbiased bases are getting more interest from researchers [1-2]. This might be because such concepts have a significant role in quantum computation and information. For instance, [3] discussed an existence of lattice structure between lines in near-linear finite geometries and its sublines. This paper establishes a relationship between topological space and the concept of non-near-linear finite geometry with variables in Z_d. The lines of this non-near-linear finite geometry are taken both through the origin $(0,0)$ and through shifted arbitrary origin(a, b) [2-3]. Previous studies focused on near-linear finite geometry. In this type of geometry, two lines have at most one point of intersection. An extension to this phenomenon is called non-near-linear geometry. It is a situation where two lines intersect at more than one point [4].

In this work, each element of the set $\{D(d)\}$ represents a finite geometry. The notation $\{D(d)\}$ represents the set of proper divisors of d. Any pairs of set of divisors form a topology in this work.

The breakdown of this work is as follows; concepts used in this work are defined in section 2, titled, preambles. Section 3 focuses on near-linear geometry. In section 4, non-near-linear finite geometry and its subgeometry is discussed. Topology and topological spaces in non-near-linear finite geometry are showcased in section 5. In section 6, the result of the findings is demonstrated using examples. The conclusion of this work is given in section 7.

2. PREAMBLES

i. The ring of integers modulo d is denoted by Z_d where$d \in Z^+$, and Z^+ represents set of positive integers. In this work, $G_d = Z_d^2$. So we use them interchangeably.

ii. $|Z^*|$ is $\phi(d)$ where Z^* represents the set of invertible element in Z_d and $\phi(d)$ is referred to as Euler Phi function. It is defined as

$$\phi(d) = \prod_{j=1}^{\ell}(p_j - 1) \quad (1)$$

iii. $\psi(d)$ is called Dedekind psi function where;

$$\psi(d) = \prod_{j=1}^{\ell}(p_j + 1),\ p_j = prime \qquad (2)$$

Here in this work, d is expressed as products of power of an integer.

3. NEAR-LINEAR FINITE GEOMETRY

In general concept, a space $S(P, L)$ is a system of points P and line L such that every line L is a subset of P and certain axioms are satisfied.

A near linear space is an incident structure $I(P, L)$ of points P and lines L such that;

i. Any line has at-least two points.
ii. Two lines meet in at most one point.

In this work, a near-linear space is defined as follows:

$$G_d = (L_d, P_d)$$

Where, P_d represents points on the line L_d.

L_d denotes lines with point P_d, where

$$L_d = \{\alpha a, \alpha b | a, b \epsilon\ Z_d, \alpha \epsilon\ Z_d\} \qquad (1)$$

Lemma: Two distinct lines of a near-linear finite geometry meet in at-most one point.

Proof:

Let $G_d = Z_d^2$

$Z_d^2 = Z_d \times Z_d$ represents lines with points in G_d. For d a prime, intersection of any pair of arbitrary lines yields a point. Hence confirm the lemma.

4. NON-NEAR-LINEAR GEOMETRY AND ITS SUB-GEOMETRY WITH VARIABLES IN Z_d

This subsection discusses the concept of finite geometry. Here two lines in a phase-space Z_d^2 meet in at least one point. Equation (1) discusses a line through the origin (0,0). This concept was discussed in 2. Shifted origin is introduced and investigated in this work. In it a line through an arbitrary point ϑ, s is named as a shifted origin. It is defined as follows:

$$L_d = \{\alpha a + \vartheta, \alpha b + s | a, b, \vartheta, s\ \epsilon\ Z_d, \}, \alpha \epsilon\ Z_d \qquad (2)$$

$\alpha \epsilon\ Z_d$ is a cyclic module over a ring of integer modulo d.

Mathematically, it is defined as the pair (P_d, L_d) in $G_d = Z_d^2$. Here,

P_d represents points in a line and L_d represents lines in G_d where,

$$P_d = \{(e, f) | e, f \in Z_d\} \qquad (3)$$

From some results obtained, we confirm the following propositions.

1. If $b \in Z_d^*$ then $L(\alpha, \beta) = L(b\alpha, b\beta)$

Now, Z_d^* represents the set of invertible elements in Z_d

Also, if $b \in Z_d - Z_d^*$ then $L(\alpha, \beta)\, mod(d) \subset L(b\alpha, b\beta)$ (4)

Hence $L(b\alpha, b\beta) \prec L(\alpha, \beta)$, where $\prec$ represents partial ordering.

We confirm that $L(\alpha, \beta)$ is a maximal line in G_d if $GCD(\alpha, \beta) \in Z_d^*$ and $L(\alpha, \beta)$ is a subline in G_d if $GCD(\alpha, \beta) \in Z_d - Z_d^*$

2. Suppose we define a line in the finite geometry G_d as in the equation (3)

   Now $L(\alpha, \beta)$ can also be;

   $$L(s\alpha, s\beta) = \{(s\xi\alpha, s\xi\beta) | \xi \in Z_d\} \xi \in Z_{\xi d} \quad (5)$$

   at the same time the line $L(\xi\alpha, \xi\beta)$ in $G_{\xi d}$ is a subline of

   $$L(\alpha, \beta) = \{(s'\alpha, s'\beta) | s' = 0, \dots, \xi d - 1\}$$

3. If two maximal lines have q points in common$q|d$.

   The q points give a subline $L(\alpha, \beta)$ where $\alpha, \beta \in \frac{d}{q} Z_q$.

If we consider the sub-geometryG_q, the subline $L(\alpha, \beta)$ in G_d is a maximal line in G_q. There is a $\psi(d)$ maximal line in sub-geometry G_q of finite geometryG_d.

The ring of integers Z_d and the Cartesian products, that is $Z_d \times Z_d$ is used extensively in this work to form finite geometry where all the lines are derived. The lines under this geometry form a non-near-linear finite geometry.

$$G_d = Z_d^2 = Z_d \times Z_d \quad (6)$$

5. TOPOLOGY AND TOPOLOGICAL SPACE ON NON-NEAR-LINEAR FINITE GEOMETRY

A topological space is a set endowed with a structure called a topology, which allows defining continuous deformation of subspaces, and more generally, all kinds of continuity [5].

This section demonstrates how a phase-space finite geometry forms a topological space with its subsets as topology. This is discussed further thus:

Definitions V.I: A set X together with the family of its subset τ is a topological space if fulfils the following conditions:

i. The empty set and the whole set are elements of τ, that is, $\varphi, X \in \tau$
ii. The union of any finite member of $\tau$is also an element of τ
iii. The intersection of any finite member of $\tau$is also an element of τ

Definitions V.II: A set X together with the topologyτ, that is (X, τ) is a topological space.

Hence in this work, a non-near-linear geometry with variables in Z_d where d is a non-prime integer is a phase-space that forms a topological space. The ring of integers modulo d is considered to be the set X, while the geometric combination $G_d = Z_d^2 = Z_d \times Z_d$ is taken as the topology. Thus, $(Z_d, Z_d \times Z_d)$ is a topological space. This phenomenon is shown to exist both when the geometric lines are taken through any arbitrary points in the geometry as defined in equations 3 and 4.

5. EXAMPLES

(a) Taking the geometry $G_{10} = Z_{10}^2$

In this example, we want to show how the finite geometric space $G_{10} = Z_{10}^2$ forms a topological space. We take the origin of the geometric lines from $(0,0)$

(a) Taking the geometry $G_{10} = Z_{10}^2$, lines of the geometry are shown thus

L(0,0) is distinct

L(0,1) ={(0,0)(0,1)(0,2)(0,3)(0,4)(0,5)(0,6)(0,7)(0,8)(0,9)}

L(0,2) ={(0,0)(0,2)(0,4)(0,6)(0,8)(0,0)(0,2)(0,4)(0,6)(0,8)}

L(0,5) is distinct

L(1,0) ={(0,0)(1,0)(2,0)(3,0)(4,0)(5,0)(6,0)(7,0)(8,0)(9,0)}

L(1,1)={(0,0)(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)(7,7)(8,8)(9,9)}

L(1,2) ={(0,0)(1,2)(2,4)(3,6)(4,8)(5,0)(6,2)(7,4)(8,6)(9,8)}

L(1,3) ={(0,0)(1,3)(2,6)(3,9)(4,2)(5,5)(6,8)(7,1)(8,4)(9,7)}

L(1,4) ={(0,0)(1,4)(2,8)(3,2)(4,6)(5,0)(6,4)(7,8)(8,2)(9,6)}

L(1,5) ={(0,0)(1,5)(2,0)(3,5)(4,0)(5,5)(6,0)(7,5)(8,0)(9,5)}

L(1,6) ={(0,0)(1,6)(2,2)(3,8)(4,4)(5,0)(6,6)(7,2)(8,8)(9,4)}

L(1,7) ={(0,0)(1,7)(2,4)(3,1)(4,8)(5,5)(6,2)(7,9)(8,6)(9,3)}

L(1,8) ={(0,0)(1,8)(2,6)(3,4)(4,2)(5,0)(6,8)(7,6)(8,4)(9,2)}

L(1,9) ={(0,0)(1,9)(2,8)(3,7)(4,6)(5,5)(6,4)(7,3)(8,2)(9,1)}

L(2,0) ={(0,0)(2,0)(4,0)(6,0)(8,0)(0,0)(2,0)(4,0)(6,0)(8,0)}

L(2,1) ={(0,0)(2,1)(4,2)(6,3)(8,4)(0,5)(2,6)(4,7)(6,8)(8,9)}

L(2,2) ={(0,0)(2,2)(4,4)(6,6)(8,8)(0,0)(2,2)(4,4)(6,6)(8,8)}

L(2,3) ={(0,0)(2,3)(4,6)(6,9)(8,2)(0,5)(2,8)(4,1)(6,4)(8,7)}

L(2,4) ={(0,0)(2,4)(4,8)(6,2)(8,6)(0,0)(2,4)(4,8)(6,2)(8,6)}

L(2,5)= {(0,0)(2,5)(4,0)(6,5)(8,0)(0,5)(2,0)(4,5)(6,0)(8,5)}

L(2,6) ={(0,0)(2,6)(4,2)(6,8)(8,4)(0,0)(2,6)(4,2)(6,8)(8,4)}

L(2,7) ={(0,0)(2,7)(4,4)(6,1)(8,8)(0,5)(2,2)(4,9)(6,6)(8,3)}

L(2,8) ={(0,0)(2,8)(4,6)(6,4)(8,2)(0,0)(2,8)(4,6)(6,4)(8,2)}

L(2,9) ={(0,0)(2,9)(4,8)(6,7)(8,6)(0,5)(2,4)(4,3)(6,2)(8,1)}

L(5,0) is distinct

L(5,1) ={(0,0)(5,1)(0,2)(5,3)(0,4)(5,5)(0,6)(5,7)(0,8)(5,9)}

L(5,2) ={(0,0)(5,2)(0,4)(5,6)(0,8)(5,0)(0,2)(5,4)(0,6)(5,8)}

L(5,5) is distinct

The following results were generated from equation (2)

$$L(\alpha,\beta) = \{(s\alpha, s\beta)|\alpha,\beta \in Z_d\}s \in Z_d$$

$$L(0,1) \cong L(0,3) \cong L(0,7) \cong L(0,9)$$

$$L(1,0) \cong L(3,0) \cong L(7,0) \cong L(9,0)$$

$$L(1,1) \cong L(3,3) \cong L(7,7) \cong L(9,9)$$

$$L(1,2) \cong L(3,6) \cong L(7,4) \cong L(9,8)$$

$$L(1,3) \cong L(3,9) \cong L(7,1) \cong L(9,7)$$

$$L(1,4) \cong L(3,8) \cong L(7,2) \cong L(9,6)$$

$$L(1,5) \cong L(3,5) \cong L(7,5) \cong L(9,5)$$

$$L(1,6) \cong L(3,8) \cong L(7,2) \cong L(9,4)$$

$$L(1,7) \cong L(3,1) \cong L(7,9) \cong L(9,3)$$

$$L(1,8) \cong L(3,4) \cong L(7,6) \cong L(9,2)$$

$$L(1,9) \cong L(3,7) \cong L(7,3) \cong L(9,1)$$

$$L(2,1) \cong L(6,3) \cong L(4,7) \cong L(8,9)$$

$$L(2,3) \cong L(6,9) \cong L(4,1) \cong L(8,7)$$

$$L(2,5) \cong L(6,5) \cong L(4,5) \cong L(8,5)$$

$$L(2,7) \cong L(6,1) \cong L(4,9) \cong L(8,3)$$

$$L(0,2) \cong L(0,4) \cong L(0,6) \cong L(0,8)$$

$$L(2,0) \cong L(4,0) \cong L(6,0) \cong L(8,0)$$

$$L(2,2) \cong L(4,4) \cong L(6,6) \cong L(8,8)$$

$$L(2,4) \cong L(4,8) \cong L(6,2) \cong L(8,6)$$

$$L(2,6) \cong L(4,2) \cong L(6,8) \cong L(8,4)$$

$$L(2,8) \cong L(4,6) \cong L(6,4) \cong L(8,2)$$

$$L(0,5), \quad L(5,0), \quad L(5,5)$$

Checking for topological space using the axioms of topology and topological space

Axiom 1:

$\varphi, X \in \tau$, here $X = Z_{10}$, $\tau = Z_{10} \times Z_{10}$.

Clearly the empty set φ is an element of the topology. That is $\varphi = L(0,0) \in \tau = Z_{10} \times Z_{10}$

Again, the whole set X is an element of τ

Hence Axiom 1 is satisfied.

Axiom 2:

Finite union of subset of $Z_{10} \times Z_{10}$is also an element of $Z_{10} \times Z_{10}$.

Clearly the union of any finite subset of $Z_{10} \times Z_{10}$ is a member element of $Z_{10} \times Z_{10}$. That is;

i. $L(1,1) \cup L(2,5) \cup L(8,2) \in Z_{10} \times Z_{10}$
ii. $L(3,4) \cup L(5,6) \in Z_{10} \times Z_{10}$
iii. $L(0,5) \cup L(5,0) \cup L(5,5) = \{(0,0)(0,5)(5,0)(5,5)\} \in Z_{10} \times Z_{10}$

Hence Axiom 2 is satisfied.

Axiom 3

Finite intersection of elements of $Z_{10} \times Z_{10}$ is again an element of $Z_{10} \times Z_{10}$

Clearly the finite intersection of elements of $Z_{10} \times Z_{10}$ is an element of $Z_{10} \times Z_{10}$

i. $L(1,1) \cap L(2,5) = (0,0) \in Z_{10} \times Z_{10}$
ii. $L(3,4) \cap L(5,6) = \{(0,0), (5,0)\} \in Z_{10} \times Z_{10}$

Hence Axiom 3 is also satisfied.

Thus, we conclude that the geometric combination $Z_{10} \times Z_{10}$ is a topology, and the combination $(Z_{10}, Z_{10} \times Z_{10})$ forms a topological space.

(b) For a shifted origin say a line through point (2,3), we check for topological space using the axioms of topology and topological space thus.

Axiom 1:

$\varphi, X \in \tau$, here $X = Z_{10}$, $\tau = Z_{10} \times Z_{10}$.

Clearly the empty set φ is an element of the topology. That is $\varphi = L(2,3) \in \tau = Z_{10} \times Z_{10}$

Again, the whole set X is an element of τ

Hence axiom 1 is satisfied.

Axiom 2

Finite union of subset of $Z_{10} \times Z_{10}$is also an element of $Z_{10} \times Z_{10}$.

Clearly the union of any finite subset of $Z_{10} \times Z_{10}$ is a member element of $Z_{10} \times Z_{10}$. That is;

i. $L(1,1) \cup L(2,5) \cup L(8,2) \in Z_{10} \times Z_{10}$
ii. $L(3,4) \cup L(5,6) \in Z_{10} \times Z_{10}$
iii. $L(0,5) \cup L(5,0) \cup L(5,5) = \{(2,3)(2,8)(7,3)(7,8)\} \in Z_{10} \times Z_{10}$

Hence axiom 2 is satisfied

Axiom 3

Finite intersection of elements of $Z_{10} \times Z_{10}$ is again an element of $Z_{10} \times Z_{10}$

Clearly the finite intersection of elements of $Z_{10} \times Z_{10}$ is an element of $Z_{10} \times Z_{10}$

i. $L(1,1) \cap L(2,5) = (2,3) \in Z_{10} \times Z_{10}$
ii. $L(3,4) \cap L(5,6) = \{(2,3), (7,3)\} \in Z_{10} \times Z_{10}$

Hence axiom 3 is satisfied.

Thus, we conclude that the geometric combination $Z_{10} \times Z_{10}$ is a topology, and the combination $(Z_{10}, Z_{10} \times Z_{10})$ forms a topological space, taking the shifted arbitrary origin.

7 CONCLUSION

Lines in finite geometry were studied. This paper focused on the relationships within subgeometries of a finite geometry. As an extension of our previous work in 18, lines in finite geometry were defined about arbitrary points. It was named a shifted origin. Our findings confirmed an existence of topology by taking a set which in our work represents points in a finite geometry and collection of all its subgeometries as the subset.

REFERENCES

[1] Vourdas, A. (2004) Quantum systems with finite Hilbert space. Report on progress in physics, 67, 1. https://doi.org/10.1088/0034-4885/67/3/R03.

[2] Shalaby, M. and Vourdas, A. (2013) Mutually Unbiased projectors and Duality between lines and Bases in Finite Quantum systems. *Annals of physics,337,208-220.*https://doi.org/10.1016/j.aop.2013.06.018

[3] S. O. Oladejo, A. D. Adeshola, and A.D Adeniyi (2019), Lattice Theory for Finite Quantum Systems, Journal of Quantum Information Science.

[4] Batten, L.M. (1997) Combinatorics of Finite Geometries. Cambridge University press, Cambridge.

[5] Bert Mendelson (1976),Introduction to Topology, Dover Publications Incorporation, New York.

[6] Albouy, O. (2009) The Isotropic lines ofZ_d^2. *Journal of physics A: Mathematical and Theoretical*, 42, Article ID:072001. https://doi.org/10.1088/1751-8113/42/7/072001.

[7] Birkhoff, Garrett, (1933) "On the Combination of Subalgebras", Proc. Cambridge,441â"464.

[8] Birkhoff, Garrett, (1967) Lattice Theory. New York.

[9] Birkhoff, Garrett, (1970) "What Can Lattices Do For You?" In: Abbott, James C. Trends in Lattice Theory. New York.

[10] Cotfas, N. and Gazeau, J.P. (2010) Quantum system with Finite Hilbert space and Frame Quantization, *Journal of physics A*, 43, Article ID:193001.

[11] Dilworth, R.P., (1961), Lattice Theory. Proceedings of Symposia in Pure Mathematics, vol. II, AMS,

[12] Francis Campe*n* ˜a (1987) Combinatorics of Finite Geometries Mathematics and Statistics Department De La Salle University.

[13] Gr*ä*tzer, George, (1971) Lattice Theory: first concepts and distributive lattices.Freeman, San Francisco,.

[14] Gr*ä*tzer, George, (1978) General Lattice Theory. Birkh*a* ¨user, Basel.

[15] Gr*ä*tzer, George, (1998) General Lattice Theory. Birkh*a* ¨user Verlag: Basel, Boston, Berlin,.

[16] Hirchfeld, J.W.P (1948) Projective Geometries over Finite Fields. Oxford University press, Oxford. 39

[17] Mehrtens, Herbert, (1979) Die Entstehung der Verbandstheorie. Gerstenberg Verlagâ"Hildesheim.

[18] Oladejo, S.O, Lei, C. and Vourdas, A. (2014) Partial Ordering of Weak Mutually Unbiased Bases. *Journal of physics A: Mathematical and Theoretical,47,* Article ID:485204. https://doi.org/10.1088/1751-8113/47/48/485204.

[19] Rota, Gian-Carlo, (1997), The Many Lives of Lattice Theory. Notices of the AMS, vol.44, no.11, 1440â"1445.

[20] Roman Schubert (2012) Linear Algebra & Geometry School of Mathematics, University of Bristol.

[21] Apostolos Vourdas (2017), Finite and Profinite Quantum Systems. Quantum Science and Technology. Springer International Publishing Switzerland.