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Numerical Methods for Solving First-Order Nonlinear Differential Equations Using a Linear Block Approach

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Abstract
This study presents a new block hybrid method aimed at solving nonlinear first-order initial value problems (IVPs). Nonlinear IVPs often appear in complex systems involving phenomena such as population dynamics, fluid flow and chemical reactions. These problems are characterized by chaotic behavior, multiple solutions and sensitivity to initial conditions, making analytical solutions difficult to obtain. To address this challenge, the study develops a block hybrid method using a linear block approach. The method employs polynomial interpolation to derive a continuous scheme, which is then discretized to generate the block method. The method is analyzed to determine its properties, including consistency, zero-stability and convergence. The block method is found to have an order of seven and the corresponding error constant is computed to demonstrate its high accuracy. A stability function is derived to examine the method's behavior and the region of absolute stability is plotted to verify its performance for stiff equations. Numerical experiments are conducted to validate the method, with results compared to existing techniques. The new block hybrid method exhibits superior accuracy, efficiency and stability when applied to nonlinear first-order IVPs, outperforming traditional methods in terms of reduced error and computational effort. This method offers a valuable alternative for solving complex nonlinear differential equations.

Keywords: Block hybrid method, Convergence analysis, Linear block approach, Nonlinear first-order IVPs, Zero-stability.

1 Introduction
Mathematics plays a crucial role in addressing empirical problems across applied sciences and various other disciplines, especially when noise is introduced into deterministic models based on differential equations [1, 2]. Traditional nonlinear differential equations have been found insufficient and ineffective for managing complex systems that involve millions or even billions of interacting particles in these areas [3, 4]. To address this, empirical problem computations will be applied, using a first-order nonlinear differential equation of the form

$$y'(x) = xy^2 + y^3 \tag{1.1}$$

These equations are more complex than their linear counterparts, as they can exhibit a wide range of behaviors, such as multiple solutions, chaos and sensitivity to initial conditions [5]. Nonlinear differential equations are used to model various real-world phenomena, including population dynamics, fluid flow, electrical circuits and chemical reactions. Solving such equations often requires specialized numerical methods, as analytical solutions are typically difficult or impossible to obtain [6-8].

Numerical Analysts and Mathematicians have long been interested in the solutions to differential equations, as well as the physical phenomena they represent. Various numerical methods have been developed for solving first-order initial value problems of ordinary differential equations, with ongoing efforts to extend these methods to higher-order equations. Several notable studies in this area include works by [9-11], among many others. These methods often employ interpolation and collocation techniques to derive solutions of varying accuracy for first-order initial value problems. The study of ordinary differential equations, including their properties and solvability, remains a highly active field of research [12-14].

2. Mathematical Formulation of the Method

This section showed the formulation of the method. The method is derived using the linear block approach [15].

The linear block approach is being of the form

$$y_{n+\xi} = \sum_{i=0}^{2} \frac{(\zeta h)^i}{i!} y_n^{(i)} + \sum_{i=0}^{4} \left(\varphi_{i\zeta} f_{n+i}\right), \quad \zeta = 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \tag{2.1}$$

is consider one after the others to obtain the method.
Differentiating of (2.1) once to obtain the block approach as

$$y_{n+\xi}^{(a)} = \sum_{i=0}^{1-a} \frac{(\zeta h)^i}{i!} y_n^{(i+a)} + \sum_{i=0}^{4} \sigma_{\zeta i a} f_{n+i}, \; a = 1_{\left(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right)} \tag{2.2}$$

$\varphi_{\zeta i} = \Omega^{-1}\mathbf{M}$ and $\sigma_{\zeta i a} = \Omega^{-1}\mathrm{N}$ where

$$\Omega = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{\left(\frac{1}{6}\right)^1}{1!} & \frac{\left(\frac{1}{3}\right)^1}{1!} & \frac{\left(\frac{1}{2}\right)^1}{1!} & \frac{\left(\frac{2}{3}\right)^1}{1!} & \frac{\left(\frac{5}{6}\right)^1}{1!} & \frac{(1)^1}{1!} \\ 0 & \frac{\left(\frac{1}{6}\right)^2}{2!} & \frac{\left(\frac{1}{3}\right)^2}{2!} & \frac{\left(\frac{1}{2}\right)^2}{2!} & \frac{\left(\frac{2}{3}\right)^2}{2!} & \frac{\left(\frac{5}{6}\right)^2}{2!} & \frac{(1)^2}{2!} \\ 0 & \frac{\left(\frac{1}{6}\right)^3}{3!} & \frac{\left(\frac{1}{3}\right)^3}{3!} & \frac{\left(\frac{1}{2}\right)^3}{3!} & \frac{\left(\frac{2}{3}\right)^3}{3!} & \frac{\left(\frac{5}{6}\right)^3}{3!} & \frac{(1)^3}{3!} \\ 0 & \frac{\left(\frac{1}{6}\right)^4}{4!} & \frac{\left(\frac{1}{3}\right)^4}{4!} & \frac{\left(\frac{1}{2}\right)^4}{4!} & \frac{\left(\frac{2}{3}\right)^4}{4!} & \frac{\left(\frac{5}{6}\right)^4}{4!} & \frac{(1)^4}{4!} \\ 0 & \frac{\left(\frac{1}{6}\right)^5}{5!} & \frac{\left(\frac{1}{3}\right)^5}{5!} & \frac{\left(\frac{1}{2}\right)^5}{5!} & \frac{\left(\frac{2}{3}\right)^5}{5!} & \frac{\left(\frac{5}{6}\right)^5}{5!} & \frac{(1)^5}{5!} \\ 0 & \frac{\left(\frac{1}{6}\right)^6}{6!} & \frac{\left(\frac{1}{3}\right)^6}{6!} & \frac{\left(\frac{1}{2}\right)^6}{6!} & \frac{\left(\frac{2}{3}\right)^6}{6!} & \frac{\left(\frac{5}{6}\right)^6}{6!} & \frac{(1)^6}{6!} \end{pmatrix}, \mathrm{M} = \begin{pmatrix} \frac{(\xi h)^1}{1!} \\ \frac{(\xi h)^2}{2!} \\ \frac{(\xi h)^3}{3!} \\ \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \frac{(\xi h)^7}{7!} \end{pmatrix}, \mathrm{N} = \begin{pmatrix} \frac{(\xi h)^{1-a}}{(1-a)!} \\ \frac{(\xi h)^{1-a}}{(1-a)!} \\ \frac{(\xi h)^{1-a}}{(1-a)!} \\ \frac{(\xi h)^{6-1}}{(1-a)!} \\ \frac{(\xi h)^{1-a}}{(1-a)!} \\ \frac{(\xi h)^{1-a}}{(1-a)!} \\ \frac{(\xi h)^{1-a}}{(1-a)!} \end{pmatrix} \tag{2.3}$$

Equation (2.1) and (2.2) are solved step by step through

$\vartheta_{\zeta}, \zeta = 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1.$

The polynomial $x = x_s + th$, is being used in (2.3) to yield the continuous scheme of the form

$$q(x_s th) = \alpha_1 y_{s+1} + h\left(\beta_0 f_s + \beta_{\frac{1}{6}} f_{s+\frac{1}{6}} + \beta_{\frac{1}{3}} f_{s+\frac{1}{3}} + \beta_{\frac{1}{2}} f_{s+\frac{1}{2}} + \beta_{\frac{2}{3}} f_{s+\frac{2}{3}} + \beta_{\frac{5}{6}} f_{s+\frac{5}{6}} + \beta_1 f_{s+1}\right) \tag{2.4}$$

Where

$\alpha_1 = 1$

$$\beta_0=-\frac{(t-1)^2\begin{pmatrix}7a+7b+7d-12t+7e+21t^2e+28t^3e+140t^4e+21at^2+28at^3+140at^4+21bt^2+28bt^3+140vt^4\\+21dt^2+28dt^3+140dt^4-7ae-7be-7de+14te-7ab-7ad-7bd+14at+14bt+14dt-18t^2\\-24t^3-30t^4-120t^5-21at^2e-168at^3e-21bt^2e-168bt^3e-21dt^2e-168dt^3e-21abt^2-168abt^3\\-21adt^2-168adt^3-21bdt^2-168bdt^3-14ate-14bte-14dte-14abt-14adt-14bdt+210abt^ee\\+210adt^2e210bdt^2e+210abdt^2+70abde-280abdte-6\end{pmatrix}}{420abde}$$

$$\beta_a=\frac{(t-1)^2\begin{pmatrix}-7b-7d+12t-7e-21t^2e-28t^3e-140t^4e-21bt^2-28bt^3-140bt^4-21dt^2-28bt^3-140dt^4\\+7dt^2+7de-14te+7bd-14bt-14bt+18t^2+24t^3+30t^4+120t^5+21bt^2e+168bt^3e+21dt^2e\\+168dt^3e+21bt^2+168bdt^3+21bdt^2+168bdt^3+14bte+14dte-14bdt-210abt^ee-6\end{pmatrix}}{420a(2a-1)(a-1)(a-e)(a-d)(a-b)}$$

$$\beta_b=-\frac{(t-1)^2\begin{pmatrix}-7a-7b+12t-7e-21t^2e-28t^3e-140t^4e-21at^2-28at^3-140at^4-21dt^2-28dt^3-140dt^4\\+7dt^4+7ae+7de+7ad-14te-14at-14dt+18t^2+24t^3+30t^4+120t^5+21at^2e+168at^3e+21dt^2e\\+168dt^3e+21adt^2+168adt^3+14ate+14dte+14dte+14adte-210abt^ee+6\end{pmatrix}}{420b(2b-1)(b-1)(b-e)(b-d)(a-b)}$$

$$\beta_{\frac{1}{2}}=-\frac{16(t-1)^2\begin{pmatrix}14a+14b+14d+20t+14e+42t^2e+56t^3e+70t^4e+42at^2+56at^3+70at^4+42bt^2+56bt^3+70bt^4\\+42dt^2 56dt^3+70dt^4-21ae-21be-21de+28te-21ab-21ad-21bd+28at+28bt+28dt-30t^2\\-40t^3-50t^4-60t^5-63at^3e-84at^3e-63bt^2e-84bt^3e-63dt^2e-84dt^3e-63abt^2-84abt^3-63adt^2\\-84adt^3-63bdt^2-84bdt^3+35abe+35ade+35bde-42ate-42bte-42dte+35abd-42abt-42adt\\-42bdt+105abt^ee+105adt^2e+105bdt^2e+105abdt^2-70abde+70abte+70adte+70bdte+70abdt\\-140abdte-10\end{pmatrix}}{105(2e-1)(2d-1)(2b-1)(2b-1)(2a-1)}$$

$$\beta_d=-\frac{(t-1)^2\begin{pmatrix}-7a-7b+12t-7e-21t^2e-28t^3e-140t^4e-21at^2-28at^3-140at^4-21bt^2-28bt^3-140bt^4\\+7ae+7be+7ab-14te-14at-14bt+18t^2+24t^3+30t^4+120t^5+21at^2e+168at^3e+21bt^2e\\+168bt^3e+21abt^2+168abt^3+14ate+14bte+14bte+14abt-210abet^2+6\end{pmatrix}}{420d(2d-1)(d-1)(d-e)(b-d)(a-d)}$$

$$\beta_e=-\frac{(t-1)^2\begin{pmatrix}7a+7b+7d-12t+21at^2+28at^3+140at^4+21bt^2+28bt^3+140bt^4+21dt^2+28dt^3+140dt^4\\-7ab-7ad-7bd+14at-7bt-7dt-18t^2-24t^3-30t^4-120t^5-21abt^2-168at^3e-21bt^2e\\-168abt^3-21adt^2-168adt^3-21bdt^2-168bdt^3-14abt-14adt-14bdt+210abdt^2-6\end{pmatrix}}{420\begin{pmatrix}-e^4+3e^5-2e^6+ae^3-3ae^4+2ae^5+be^3-3be^4+2be^5+de^3+3de^4+2de^5-abe^2+3abe^3\\-2abe^4+ade^2+3ade^3-2ade^4-bde^2+3bde^3-2bde^4+abde-3abde^2+2abde^3\end{pmatrix}}$$

$$\beta_1=-\frac{(t-1)\begin{pmatrix}56a+56b+56d-56t+56e+56t^2e+56t^3e+56t^4e++56t^5e++56at^2+56at^3+56at^4+56bt^2\\140at^5+56bt^3+56bt^4+140bt^5+56dt^2+56dt^3+56dt^4+140dt^5-63ae-63be-63de+56te\\-63ab-63ad-63bd+56at+56bt+56dt-50t^2-50t^3-50t^4-50t^5-120t^6-63att^2e\\-63at^3e-160att^4e-63bt^2e-63bt^3e-168bt^4e-63dt^2e-63dt^3e-168dt^4e-63abt^2-63abt^3\\168abt^4-63adt^2-63adt^3-168adt^4-63bdt^2-63bdt^3-168bdt^4+70abe+70ade+70bde\\-63ate-63bte-63dte+70abd-63abt-63adt-63bdt+70abt^2e+210abt^3e+70bdt^2e+\\210bdt^3e+70abdt^2+210abdt^3-70abde+70abte+70adte-280abdt^2e-70abdte-50\end{pmatrix}}{420((-e+1)(d-1)(a-1))}$$

To get the unknown values of (2.3), we simplify $\sigma_{\zeta i a}=\Omega^{-1}\mathrm{N}$ to obtain

$$
\left.
\begin{aligned}
y_{n+a} &= y_n + h\left(\sigma_{115} f_n + \sigma_{120} f_{n+a} + \sigma_{125} f_{n+b} + \sigma_{130} f_{n+\frac{1}{2}} + \sigma_{135} f_{n+d} + \sigma_{140} f_{n+e} + \sigma_{145} f_{n+1}\right) \\
y_{n+b} &= y_n + h\left(\sigma_{215} f_n + \sigma_{220} f_{n+a} + \sigma_{225} f_{n+b} + \sigma_{230} f_{n+\frac{1}{2}} + \sigma_{235} f_{n+d} + \sigma_{240} f_{n+e} + \sigma_{245} f_{n+1}\right) \\
y_{n+\frac{1}{2}} &= y_n + h\left(\sigma_{315} f_n + \sigma_{320} f_{n+a} + \sigma_{325} f_{n+b} + \sigma_{330} f_{n+\frac{1}{2}} + \sigma_{335} f_{n+d} + \sigma_{340} f_{n+e} + \sigma_{345} f_{n+1}\right) \\
y_{n+d} &= y_n + h\left(\sigma_{415} f_n + \sigma_{420} f_{n+a} + \sigma_{425} f_{n+b} + \sigma_{430} f_{n+\frac{1}{2}} + \sigma_{435} f_{n+d} + \sigma_{440} f_{n+e} + \sigma_{445} f_{n+1}\right) \\
y_{n+e} &= y_n + h\left(\sigma_{515} f_n + \sigma_{520} f_{n+a} + \sigma_{525} f_{n+b} + \sigma_{530} f_{n+\frac{1}{2}} + \sigma_{535} f_{n+d} + \sigma_{540} f_{n+e} + \sigma_{545} f_{n+1}\right) \\
y_{n+1} &= y_n + h\left(\sigma_{615} f_n + \sigma_{620} f_{n+a} + \sigma_{625} f_{n+b} + \sigma_{630} f_{n+\frac{1}{2}} + \sigma_{635} f_{n+d} + \sigma_{640} f_{n+e} + \sigma_{645} f_{n+1}\right)
\end{aligned}
\right\} \qquad (2.5)
$$

Where

$$
\begin{pmatrix} \sigma_{115} \\ \sigma_{120} \\ \sigma_{125} \\ \sigma_{130} \\ \sigma_{135} \\ \sigma_{140} \\ \sigma_{145} \end{pmatrix} = \begin{pmatrix} \frac{19089}{362880} \\ \frac{2713}{15120} \\ -\frac{15487}{120960} \\ \frac{293}{2835} \\ -\frac{6737}{120960} \\ \frac{263}{15120} \\ -\frac{863}{362880} \end{pmatrix},
\begin{pmatrix} \sigma_{215} \\ \sigma_{220} \\ \sigma_{225} \\ \sigma_{230} \\ \sigma_{235} \\ \sigma_{240} \\ \sigma_{245} \end{pmatrix} = \begin{pmatrix} \frac{1139}{22680} \\ \frac{47}{189} \\ \frac{11}{7560} \\ \frac{166}{2835} \\ -\frac{269}{7560} \\ \frac{11}{945} \\ -\frac{37}{22680} \end{pmatrix},
\begin{pmatrix} \sigma_{315} \\ \sigma_{320} \\ \sigma_{325} \\ \sigma_{330} \\ \sigma_{335} \\ \sigma_{340} \\ \sigma_{135} \end{pmatrix} = \begin{pmatrix} \frac{137}{2688} \\ \frac{27}{112} \\ \frac{387}{4480} \\ \frac{17}{105} \\ -\frac{243}{4480} \\ \frac{9}{560} \\ -\frac{29}{13440} \end{pmatrix},
\begin{pmatrix} \sigma_{415} \\ \sigma_{420} \\ \sigma_{425} \\ \sigma_{430} \\ \sigma_{435} \\ \sigma_{440} \\ \sigma_{445} \end{pmatrix} = \begin{pmatrix} \frac{143}{2835} \\ \frac{232}{945} \\ \frac{64}{945} \\ \frac{752}{2835} \\ \frac{29}{945} \\ \frac{8}{945} \\ -\frac{4}{2835} \end{pmatrix},
\begin{pmatrix} \sigma_{515} \\ \sigma_{5120} \\ \sigma_{525} \\ \sigma_{530} \\ \sigma_{535} \\ \sigma_{540} \\ \sigma_{545} \end{pmatrix} = \begin{pmatrix} \frac{3715}{72576} \\ \frac{727}{3024} \\ \frac{2125}{24192} \\ \frac{125}{567} \\ \frac{3875}{24192} \\ \frac{235}{3024} \\ -\frac{275}{72576} \end{pmatrix},
\begin{pmatrix} \sigma_{615} \\ \sigma_{620} \\ \sigma_{625} \\ \sigma_{630} \\ \sigma_{635} \\ \sigma_{640} \\ \sigma_{645} \end{pmatrix} = \begin{pmatrix} \frac{41}{840} \\ \frac{9}{35} \\ \frac{9}{280} \\ \frac{34}{105} \\ \frac{9}{280} \\ \frac{9}{35} \\ \frac{41}{840} \end{pmatrix}
$$

3 Basic Properties of the Block Method

3.1 Order and Error Constant

This subsection establishes the linear operator $\ell[y(x_i);h]$ associated with the newly derived method (2.5).

Proposition 1

The local truncation error of the newly derived scheme is $C_{07}h^{07}y^{07}(x_n)+0(h^{08})$.

Proof

The linear difference operators associated with the hybrid method (2.5) are given by [5].

$$\left.\begin{aligned}
\ell[y(x_n);h] &= y(x_n+ah)-\left(\alpha_1(x_\eta+h)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,a,b,\frac{1}{2},d,e,1 \\
\ell[y(x_n);h] &= y(x_n+bh)-\left(\alpha_1(x_\eta+h)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,a,b,\frac{1}{2},d,e,1 \\
\ell[y(x_n);h] &= y\left(x_n+\frac{1}{2}h\right)-\left(\alpha_1(x_\eta+h)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,a,b,\frac{1}{2},d,e,1 \\
\ell[y(x_n);h] &= y(x_n+dh)-\left(\alpha_1(x_\eta+h)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,a,b,\frac{1}{2},d,e,1 \\
\ell[y(x_n);h] &= y(x_n+eh)-\left(\alpha_1(x_\eta+h)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,a,b,\frac{1}{2},d,e,1 \\
\ell[y(x_n);h] &= y(x_n+h)-\left(\alpha_1(x_\eta+h)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,a,b,\frac{1}{2},d,e,1
\end{aligned}\right\} \quad (3.1)$$

If $y(x)$ is sufficiently differentiable, we can use the Taylor series to expand equation (3.1) in the power of h. It is critical to emphasize that the first non-zero term in each formula in Equation (3.1) is $C_{07}h^{07}y^{07}(x_n)+0(h^{08})$

Definition 1. [5]

A linear multistep method is of order p if it satisfies the condition

$c_0=c_1=c_2=c_3=\cdots=c_p=c_{p+1}=0,\ c_{p+2}\neq 0,$ where

$$\left.\begin{aligned}
c_0 &= \sum_{j=0}^{k}\alpha_j \\
c_1 &= \sum_{j=0}^{k}\left(j\alpha_j-\beta_j\right) \\
&. \\
&. \\
&. \\
c_p &= \sum_{j=0}^{k}\left[\frac{1}{p!}j^p\alpha_j-\frac{1}{(p-1)!}\left(j^{p-1}\beta_j\right)\right],\ p=2,3,\cdots,q+1
\end{aligned}\right\} \quad (3.2)$$

The parameter $c_{p+2}\neq 0$ is referred to as the error constant with the local truncation error defined as

$$x_{n+k} = c_{p+2} h^{p+2} y^{(p+2)}(x_n) + c_{p+3} h^{p+3} y^{(p+3)}(x_n) + c_{p+4} h^{p+4} y^{(p+4)}(x_n) + 0(h^{p+5}) \quad (3.3)$$

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{(a)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\sigma_{120}(a) + \sigma_{125}(b) + \sigma_{130}\left(\frac{1}{2}\right) + \sigma_{135}(d) + \sigma_{140}(e) + \sigma_{145}(1) \right] \\ \sum_{j=0}^{\infty} \frac{(b)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\sigma_{220}(a) + \sigma_{225}(b) + \sigma_{230}\left(\frac{1}{2}\right) + \sigma_{235}(d) + \sigma_{240}(e) + \sigma_{245}(1) \right] \\ \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\sigma_{320}(a) + \sigma_{325}(b) + \sigma_{330}\left(\frac{1}{2}\right) + \sigma_{335}(d) + \sigma_{340}(e) + \sigma_{345}(1) \right] \\ \sum_{j=0}^{\infty} \frac{(d)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\sigma_{420}(a) + \sigma_{425}(b) + \sigma_{430}\left(\frac{1}{2}\right) + \sigma_{435}(d) + \sigma_{440}(e) + \sigma_{445}(1) \right] \\ \sum_{j=0}^{\infty} \frac{(e)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\sigma_{520}(a) + \sigma_{525}(b) + \sigma_{530}\left(\frac{1}{2}\right) + \sigma_{535}(d) + \sigma_{540}(e) + \sigma_{545}(1) \right] \\ \sum_{j=0}^{\infty} \frac{(1)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\sigma_{620}(a) + \sigma_{625}(b) + \sigma_{630}\left(\frac{1}{2}\right) + \sigma_{635}(d) + \sigma_{640}(e) + \sigma_{645}(1) \right] \end{bmatrix} \quad (3.4)$$

Corollary 1 [5].

The newly derived method (2.5) has a local truncation error given by

$$\left.\begin{matrix} (6.7679 \times 10^{-09}) C_{07} h^{07} y^{07}(x_n) + 0(h^{08}) \\ (5.0402 \times 10^{-09}) C_{07} h^{07} y^{07}(x_n) + 0(h^{08}) \\ (5.9803 \times 10^{-09}) C_{07} h^{07} y^{07}(x_n) + 0(h^{08}) \\ (5.0402 \times 10^{-09}) C_{07} h^{07} y^{07}(x_n) + 0(h^{08}) \\ (6.7679 \times 10^{-09}) C_{07} h^{07} y^{07}(x_n) + 0(h^{08}) \\ (5.9803 \times 10^{-09}) C_{07} h^{07} y^{07}(x_n) + 0(h^{08}) \end{matrix}\right\} \quad (3.5)$$

Therefore, the newly derived scheme is of uniform order seven as well as error constant is given by

$$C_8 = \begin{pmatrix} 6.7679 \times 10^{-09} \\ 5.0402 \times 10^{-09} \\ 5.9803 \times 10^{-09} \\ 5.0402 \times 10^{-09} \\ 6.7679 \times 10^{-09} \\ 5.9803 \times 10^{-09} \end{pmatrix}$$

3.2 Consistent

Traditionally, the method is consistent because the order of the method is order greater than or equal to one.

3.3 Zero stable

Definition 2. [5]

A linear multistep method is said to be zero stable as $h \to 0$ if the roots of the polynomial $\pi(r) = 0$ satisfy $\left|\left[\sum A^0 R^{k-1}\right]\right| \le 1,$ and those roots with R = 1 must be simple.

Hence according to [5] it's found as

$$\pi(r) = \left| r \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} r & 0 & 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 & 0 & -1 \\ 0 & 0 & r & 0 & 0 & -1 \\ 0 & 0 & 0 & r & 0 & -1 \\ 0 & 0 & 0 & 0 & r & -1 \\ 0 & 0 & 0 & 0 & 0 & r-1 \end{bmatrix} \right| = r^6(r-1) \qquad (3.6)$$

Then, solving for r in $r^6(r-1)$,

gives $r = 0, 0, 0, 0, 0, 1$. Therefore, the method is zero stable.

Dahlquist's theorem states that the scheme is convergent, and consistency and zero-stability are analyzed and fulfilled [5].

3.4. Convergence

Theorem 1. [5]

Consistency and zero-stability are both required and sufficient conditions for a linear multistep method to be convergent. Therefore, the newly derived scheme is convergent since it is consistent and zero-stable.

3.5. Linear Stability

Definition 3. [5]

The region of absolute stability of a numerical method is the set of complex values λh for which all solutions of the test problem $y' = -\lambda y$ will remain bounded as $n \to \infty$.

The concept of A-stability according to [5] is discussed by applying the test equation $y^{(k)} = \lambda^{(k)} y$

To yield

$$Y_m = \mu(z) Y_{m-1}, \; z = \lambda h \qquad (3.7)$$

Where $\mu(z)$ is the amplification matrix of the form

$$\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^1\eta^{(0)}\right)^{-1}\left(\xi^1 - z\eta^{(1)} - z^1\eta^{(1)}\right) \qquad (3.8)$$

The matrix $\mu(z)$ has Eigen values $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function of the method is given by

$$\zeta = -\frac{\begin{pmatrix} 367275240z^6 - 1000075262\,8z^5 + 7978519183\,4z^4 + 5060796756\,30z^3 \\ + 1827771257\,925z^2 - 4328380929\,600z + 4444263936\,000 \end{pmatrix}}{\begin{pmatrix} 870912000z^6 + -128024064\,00z^5 + 1060770816\,00z^4 - 5761082880\,00z^3 \\ + 2057529600\,000z^2 - 4444263936\,000z + 4444263936\,000 \end{pmatrix}} \qquad (3.9)$$

The boundary locus method is used to generate the hybrid method's stability polynomial. The polynomial is defined as

$$\bar{h}(w)=\left(-\frac{1}{326592}w^5+\frac{1}{326592}w^6\right)h^6+\left(-\frac{7}{77760}w^5-\frac{7}{77760}w^6\right)h^5+\left(-\frac{29}{19440}w^5+\frac{29}{19440}w^6\right)h^4 +\left(-\frac{7}{432}w^5-\frac{7}{432}w^6\right)h^3+\left(-\frac{25}{216}w^5+\frac{25}{216}w^6\right)h^2+\left(-\frac{1}{2}w^5+\frac{1}{2}w^6\right)h-w^5+w^6 \quad (3.10)$$

The polynomial is used to plot the region as

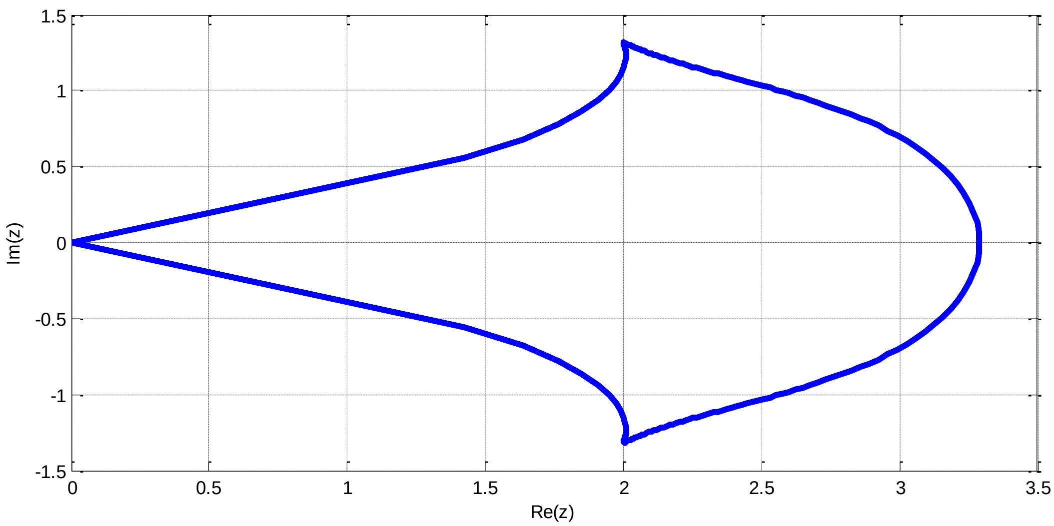

Fig. 1: Showing an *A-stable* region of absolute stability of the new Method

4.2 Numerical Experiments

The newly derived block hybrid method (2.5) shall be applied on four nonlinear first order initial value problems ordinary differential equations of the form (1.1) as presented below.

The results obtained on the application of nonlinear first order initial value problems ordinary differential equations were compared with the existing methods of [16-21].

Problem 4.1

Consider a nonlinear first order initial value problems ordinary differential problem which was solved by [16, 17] formulated as follows:

$$y'(x)=-10(y-1)^2,\ y(0)=2,\ h=0.01 \quad (4.1)$$

with the exact solution given by

$$y(x)=1+\frac{1}{1+10x} \quad (4.2)$$

Problem 4.2

Given a nonlinear first order ordinary differential problem which was addressed by [17, 18] formulated as follows:

$$y'=\frac{1}{2}(1-y),\ y(0)=\frac{1}{2}, \quad (4.3)$$

Which has the exact solution as

$$y(x)=1-\frac{1}{2}e^{-\frac{1}{2}x} \quad (4.4)$$

Problem 4.3

Given a nonlinear first order ordinary differential problem which was addressed by [19, 20] formulated as follows:

$$y'=-y^2,\ y(0)=1,\ 0\le x\le 1 \quad (4.5)$$

Whose exact solution is given as

$$y(x) = \frac{1}{1+x} \quad (4.6)$$

Problem 4.4:

Given a nonlinear first order ordinary differential problem which was addressed by [21] formulated as follows:

$$y' = 4 - 4y + y^2 = 0,\ y(0) = 1,\ 0 \le x \le 1 \quad (4.7)$$

Whose exact solution are

$$y(x) = \frac{2x-1}{x-1} \quad (4.8)$$

Table 4.2: The results of application problem 4.2 with [16, 17].

x	Exact Solution	Computed Solution	ENM	E[16]	E[17]
0.1	1.9090909090909090909	1.9090909090909082926910	7.98218e-14	1.75330e-10	1.55825e-06
0.2	1.8333333333333333333	1.8333333333333323389440	9.94389e-14	2.32000e-10	2.39975e-06
0.3	1.7692307692307692308	1.8333333333333323389440	9.88733e-14	2.41150e-10	2.83045e-06
0.4	1.7142857142857142857	1.71428571428562244730	9.18384e-14	2.31400e-10	3.02094e-06
0.5	1.6666666666666666667	1.666666666666658342770	8.32390e-14	2.14840e-10	3.06956e-06
0.6	1.6250000000000000000	1.6249999999999992517240	7.48276e-14	1.96600e-10	3.03457e-06
0.7	1.5882352941176470588	1.58823529411757987960	6.71792e-14	1.78870e-10	2.95115e-06
0.8	1.5555555555555555556	1.555555555555549513440	6.04212e-14	1.62500e-10	2.84088e-06
0.9	1.5263157894736842105	1.52631578947362969500	5.45155e-14	1.47730e-10	2.71713e-06
1.0	1.5000000000000000000	1.4999999999999995062980	4.93702e-14	1.34570e-10	2.58816e-06

Table 4.2: The results of application problem 4.2 with [17, 18]

x	Exact Solution	Computed Solution	ENM	E[18]	E[17]
0.1	0.52438528774964299546	0.52438528774964299546	0.00000e-00	3.01260e-17	1.99840e-15
0.2	0.54758129098202021342	0.54758129098202021342	0.00000e00	5.49357e-17	3.88578e-15
0.3	0.56964601178747109638	0.56964601178747109639	1.00000e-20	5.83702e-17	5.44009e-15
0.4	0.59063462346100907066	0.59063462346100907067	1.00000e-20	5.89498e-17	6.99441e-15
0.5	0.61059960846429756588	0.61059960846429756590	2.00000e-20	4.20996e-17	8.21565e-15
0.6	0.62959088965914106696	0.62959088965914106699	3.00000e-20	8.43851e-17	9.54792e-15
0.7	0.64765595514064328282	0.64765595514064328285	3.00000e-20	8.85311e-17	1.05471e-14
0.8	0.66483997698218034963	0.66483997698218034965	2.00000e-20	9.33604e-17	1.13243e-14
0.9	0.68118592418911335343	0.68118592418911335346	3.00000e-20	2.67745e-17	1.22125e-14
1.0	0.69673467014368328820	0.69673467014368328823	3.00000e-20	2.98500e-16	1.28786e-14

Table 4.3: The results of application problem 4.3 with [19, 20]

x	Exact Solution	Computed Solution	ENM	E[19]	E[20]
0.01	0.99009900990099009899	0.99009900990099009901	2.00000e-20	2:40000e-04	2.91799e-11
0.02	0.98039215686274509804	0.98039215686274509800	4.00000e-20	5.60000e-04	3.71577e-11
0.03	0.97087378640776699029	0.97087378640776699023	6.00000e-20	7.10000e-04	3.93663e-11
0.04	0.96153846153846153846	0.96153846153846153839	7.00000e-20	8.40000e-04	3.39936e-11
0.05	0.95238095238095238095	0.95238095238095238086	9.00000e-20	9.40000e-04	2.94922e-11
0.06	0.94339622641509433962	0.95238095238095238086	1.10000e-19	1.10000e-04	2.61278e-11
0.07	0.93457943925233644860	0.94339622641509433951	1.30000e-19	1.10000e-03	2.31487e-11
0.08	0.92592592592592592593	0.93457943925233644847	1.40000e-19	1.30000e-03	6.80704e-11
0.09	0.91743119266055045872	0.92592592592592592579	1.60000e-19	1.50000e-03	8.31745e-11
0.10	0.90909090909090909091	0.90909090909090909074	1.70000e-19	1.60000e-02	7.50649e-11

Table 4.4: The results of application problem 4.4 with [21].

x	Exact Solution	Computed Solution	ENM	E[21]
0.1	0.98989898989898989899	0.98989898989898989897	2.00000e-20	1.11400e-04
0.2	0.97959183673469387755	0.97959183673469387751	4.00000e-20	1.85700e-03
0.3	0.96907216494845360825	0.96907216494845360819	6.00000e-20	2.53600e-04
0.4	0.95833333333333333333	0.95833333333333333326	7.00000e-20	7.14800e-05
0.5	0.94736842105263157895	0.94736842105263157885	1.00000e-19	9.72000e-06
0.6	0.93617021276595744681	0.93617021276595744670	1.10000e-19	2.06400e-06
0.7	0.92473118279569892473	0.92473118279569892460	1.30000e-19	2.80400e-07
0.8	0.91304347826086956522	0.91304347826086956507	1.50000e-19	5.29200e-08
0.9	0.90109890109890109890	0.90109890109890109873	1.70000e-19	7.16300e-09
1.0	0.88888888888888888889	0.88888888888888888889	0.00000e00	1.09300e-09

4.3 Discussion of Results and Conclusion

In the analysis of Problems 4.1 and 4.2, the new method shows a significant improvement in accuracy over existing methods, such as those proposed by [16-18]. The computed solutions produced by the new method align closely with the exact solutions, demonstrating its precision in solving nonlinear first-order initial value problems. This accuracy is reflected in the consistently smaller absolute errors compared to the larger errors of the other methods, making the new approach a more reliable tool for precise approximations. The method's ability to maintain a minimal error margin throughout various independent variables further underscores its robustness and effectiveness in solving differential equations.

The examination of Problem 4.3 highlights the remarkable accuracy of the new method, particularly in comparison to the approaches of [19, 20]. The new method's computed solutions align closely with the exact solutions, consistently producing minimal absolute errors. In contrast, the methods by [19, 20] show larger error margins, indicating less precise approximations. The ability of the new method to maintain accuracy across the entire range of values underscores its stability and robustness. Similarly, in Problem 4.4, the new method demonstrates a clear advantage over the method proposed by [21], with its computed solutions closely matching the exact solutions and maintaining lower error margins. This precision and consistency further validate the new method as a reliable tool for solving nonlinear first-order ordinary differential equations with high accuracy.

In conclusion, the study introduces an implicit one-step block hybrid method for solving nonlinear first-order ordinary differential equations (ODEs). By employing the linear block approach, the new method enhances accuracy, stability and computational efficiency, addressing common challenges such as error propagation in nonlinear problems. The method consistently demonstrated superior performance in Problems 4.1 to 4.4, surpassing existing approaches by [16-21], with smaller absolute errors and a close alignment to exact solutions. The stability analysis further validated its robustness, making this method a highly reliable and effective tool for solving nonlinear first-order ODEs.

References

[1] Skwame, Y., Sabo, J. & Kyagya, T. Y. (2017). The constructions of implicit one-step block hybrid methods with multiple off-grid points for the solution of stiff differential equations. *Journal of Scientific Research and Report*. 16(1): 1-7.

[2] Tumba, P. and Sabo, J. (2024). The Efficiency of Block Hybrid Method for Solving Malthusian Growth Model and Prothero-Robinson Oscillatory Differential Equations. International Journal of Development Mathematics 1(3); 08-22. https://doi.org/10.62054/ijdm/0103.02.

[3] Sabo, J., Tumba, P. & Bakari, I. A. (2018). The modification of second derivative linear multistep ordinary differential equation for solving stiffly differential equation. *IOSR Journal of Mathematics.* 14, 48-55.

[4] Sabo, J., Raymond, D., Bambur, A. A. & Kyagya, T. Y. (2019). An A-stable uniform order six linear multi-step method for testing stiffly first order ordinary differential equations. *Invention Journal of Research Technology in Engineering & Management.* 3(1): 23-30.

[5] Sabo, J., Ayinde, A. M., Oyedepo, T. and Ishaq, A. A. (2024). Simulation of Two-Step Block Approach for Solving Oscillatory Differential Equations. Cumhuriyet Science Journal, 45(2), 366-378. https://doi.org/10.17776/csj.1345303.

[6] Aboiyar T., Luga T., & Iyorter B. V. (2015). Derivation of continuous linear multistep methods using hermite polynomials as basic functions," *American Journal of Applied Mathematics and Statistics*, 3:220-225.

[7] Aksah J. S., Ibrahim Z. B. and Zawawi I. S. M., 2019. "Stability analysis of singly diagonally Implicit Block Backward Differentiation Formulas for stiff ordinary differential equations", *Research gate Publication,* http://doi.org/10.3390/math7020211.

[8] Awoyemi D. O., Kayode S. J. and Adoghe L. O., 2014. "A five step P-stable method for the numerical integration of third order ordinary differential equations", *American Journal of Computational Mathematics* 4:119 – 126.

[9] Ayinde S. O. and Ibijola E. A., 2015. "A new numerical method for solving first order differential equations", *American Journal of Applied Mathematics and Statistics.* 3(4): 156-160.

[10] Fadugba S. E and Falodun B. O., 2017. "Development of a new one-step scheme for the solution of initial value problem (IVP) in ordinary differential equations", *International Journal of Theoretical and Applied Mathematics* 3:58.

[11] Fadugba S. E. and Idowu J. O., 2019. "Analysis of the properties of a third order convergence numerical method derived via transcendental function of exponential form", *International Journal of Applied Mathematics and Theoretical Physics* 5: 97-110.

[12] Fadugba S. E., Ogunyebia S. N. and Falodun B. O., 2020. "An Examination of a Second order numerical method for solving initial value problems. *Journal of the Nigerian Society of Physical Sciences, J. Nig. Soc. Phys. Sci*. 2: 120–127.

[13] Olanegan O. O., Ogunware B. G., Omole E. O., Oyinloye T. S., and Enoch B. T., 2016. Some classes of continous hybrid linear mulstistep methods for the solution of first order ordinary differential equations," *International Organization of Scientific Research*, 11: 08 - 13, Available at: 10.9790/5728-11510813.

[14] Ayinde, A. M., Ibrahim, S., Sabo, J. & Silas, D. (2023). The physical application of motion using single step block method. *Journal of Material Science Research and Review.* 6(4), 708-719.

[15] Sabo, J., Kyagya, T. Y. & Solomon, M. (2021). One-step hybrid block scheme for the numerical approximation for solution of third order initial value problems. *Journal of Scientific Research & Reports.* 27(12): 51-61.

[16] Kamoh, N., Kumleng G. M. and Sunday, J. (2021). Continuous One Step Linear Multi-Step Hybrid Block Method for the Solution of First Order Linear and Nonlinear Initial Value Problem of Ordinary Differential Equations. *Recent Developments in the Solution of Nonlinear Differential Equations. DOI: http://dx.doi.org/10.5772/intechopen.95619*.

[17] Kashkari, B. S., Syam, M. I. (2019). Optimization of one-step block method with three hybrid points for solving first order ordinary differential equation. Results in Physics 12, 592–596. https://doi.org/10.1016/j.rinp.2018.12.015.

[18] AbdulAzeez K. J. and Adebayo, O. A. (2022). A two-step block method with two hybrid points for the numerical solution of first order ordinary differential equations. Open J. Math. Sci. 6, 281-288; doi:10.30538/oms2022.0193.

[19] E. A. Areo, R. A. Ademiluyi and P. O. Babatola, Accurate Collocation Multistep method for Integration of first order ordinary Differential Equation, Int. J. Comp. Maths, 2(1) 15-27, 2008.
[20] S. A. Odejide and A. O. Adeniran (2012). A hybrid linear collocation multistep scheme for solving first order initial value problems. Journal of the Nigerian Mathematical Society. Vol. 31, pp. 229-241.
[21] Isah, I. O., Salawu, A. S., Olayemi, K. S. and Enest, L. O. (2020). An efficient 4-step block method for first order initial value problems via shifted chebyshev polynomial. Tropical Journal of Science and Technology. 1(2), 25-36. http://doi.org/10.47524/tjst.v1i2.5.